ROBUSTNESS ANALYSIS OF UNCERTAIN LINEAR SYSTEMS AND ROBUST STABILIZATION OF UNCERTAIN DELAYED SYSTEMS

By

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To my parents

Smt. Shyamala & Mr. Varadharajan Radhamani

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Chairman: Dr. Oscar D. Crisalle
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This dissertation focuses on two main aspects. One, developing new tools for the robustness analysis of uncertain linear systems. Two, the synthesis of robust controllers for uncertain delay systems.

Traditional methods to analyze the robust stability of linear systems have depended on structured singular value bounds. However, this approach neglects phase dependence of the uncertainty and results in undue conservatism. In this dissertation, the concept of the Nyquist robust-stability margin, is introduced for characterizing the closed-loop stability of uncertain systems. The approach makes direct use of Nyquist domain arguments and is based on the analysis of the perturbed eigenvalue loci, hence avoiding undue conservatism through the use of singular-value upper bounds. A key element in the new approach is Critical Direction Theory applied to uncertainties in the Nyquist plane. The critical direction method is based on recognizing that, at any given frequency on the Nyquist plane, there is only one direction of perturbation of relevance to the stability analysis. This allows the characterization of robust stability margins for

uncertain systems characterized by irregular perturbation templates, a problem that poses significant challenges to other analysis methods. Examples of practical relevance are given to illustrate the application of the new theory. Using the new approach, the problem of assessing robust stability and computing stability margins for SISO systems with affine complex parametric uncertainties is tackled successfully. Exact analytical results are derived for geometrically simple uncertainty sets such as ellipses and rectangles.

In the later part of the dissertation, the synthesis of robust controllers for state-delayed and input-delayed systems is considered. In particular, *Sliding Mode Control* is chosen as the technique of choice, as it possesses the combination of robustness and performance guarantees that one seeks in a control system. Robust stability to the chosen perturbation characterization is rigorously proven. Practical difficulties in implementation are pointed out and ways to overcome these hurdles are presented. Some open questions in the literature are brought out and theoretical analysis and analytical answers are presented. Finally, ideas and interesting formulations are presented for future work.

CHAPTER 1
INTRODUCTION

Modeling of any practical system for control design invariably involves uncertainty. Since the exact model of a process may be difficult or impossible to determine, the logical approach is to design a control strategy based on a suitable nominal (*i.e.*, most likely) model. Once a system has been identified and a nominal model and associated uncertainty description is available, there are two main tasks. The first task is the analysis of the uncertain system. The following issues arise naturally: The first, is the question of robustness analysis. What is the measure of the maximum uncertainty that the system can tolerate and still sustain stability and performance? Alternatively, given the size of the uncertainty, one might wish to study the effect of the uncertainty on the stability and performance of the system. A system is said to be robustly stable if it can sustain stability in spite of suffering from uncertainty.

The second major issue is the synthesis of robust controller to ensure stability and performance of the physical system. Over the past two decades, a plethora of robust controller methods have been investigated and implemented in a wide array of industries. Although, robust analysis and synthesis is an intellectually stimulating and mathematically challenging task, there are some valid criticisms that can be leveled against this approach. Is it a good idea to design a system for worst case uncertainty? If the probability of worst case occurring is very low, why should we sacrifice the performance of the system by designing for the worst case? Should the uncertainties be simply bounded by norms (or) should there be probabilistic ideas inherently associated with the uncertainty descriptions? These are areas of intense interest and importance and the interested reader is referred to the research monograph by Smith and Dahleh [1994]

and the references therein.

In this dissertation attention will be focused on both the analysis and the synthesis aspects of robust control. The thesis is organized as follows: Chapters 2-3 address the robustness analysis problem, while Chapters 4-5 deal with controller design for robustly stabilizing a class of uncertain delay systems. In Chapter 6, future research directions are presented.

In Chapter 2, the concept of the *Nyquist robust-stability margin*, for characterizing the closed-loop stability of uncertain systems is introduced. The predominant robustness analysis methods typically rely on arguments based on real singular value or magnitude bounds. Such is the case of well known multivariable stability margin k_m [Safonov, 1982] and the structured singular value μ [Doyle, 1982]. In contrast, the proposed new approach makes direct use of Nyquist arguments and is based on the analysis of the perturbed eigenvalue loci, hence avoiding undue conservatism that may result through the use of singular-value upper bounds. A key element in the new approach is *Critical Direction Theory* applied to uncertainties in the Nyquist plane. The critical direction method is based on recognizing that, at any given frequency on the Nyquist plane, there is only one direction of perturbation of relevance to the stability analysis. This critical direction is defined by the oriented line that has its origin at the location of the nominal eigenvalue of the unperturbed system and passes through the critical point -1+j0. In fact, from a stability point of view all points on the uncertainty eigentemplate that do not lie on the critical line can be ignored. This allows the characterization of robust stability margins for uncertain systems characterized by irregular perturbation templates, a problem that poses significant challenges to other analysis methods. Other attractive features of the Nyquist robust-stability margin theory are that its MIMO (Multiple Input-Multiple Output) form is a natural extension of the SISO (Single Input-Single Output) case, and that its versions for continuous and discrete systems are formally identical. A SISO and a MIMO example are given illustrating the concepts.

Chapter 3 focuses on the complex parametric uncertainty problem from the perspective of the *critical direction* paradigm. Considerable attention has been given to the problem of assessing robust stability and computing stability margins for SISO systems with parametric uncertainties. While the general robust stability margin computation problem is known to be NP-hard [Braatz *et al.*, 1994], its rank-one counterpart results in a convex optimization problem that can be readily computed, and can be often derived as an explicit analytical expression. It is now well known that affine parametric uncertainty problems result in a rank-one μ formulation. This chapter addresses the case of element-wise complex parametric uncertainties where each individual uncertainty lies in a highly structured domain that is convex along any line from the origin. This is a mild convexity condition, since the actual domain could be highly non-convex. Exact analytical results are derived for geometrically simple uncertainty sets such as ellipses and rectangles. This approach however, is applicable to a much wider class of systems than the ones considered here. The only information that is needed is the phase dependent magnitude of the uncertainty. The approach presented here is simple, intuitive and yet, mathematically rigorous. For the special case of disk-bounded affine uncertainties, the approach recovers, in a transparent manner, relevant results by previous methods.

Chapter 4 and Chapter 5 deal with the issue of stabilizing an uncertain time-delay systems (both state-delay and input-delay are considered). Time delay exists in various branches of engineering. From biological, optical, electric networks, to chemical reaction systems, time delay occurs and affects the stability and performance of a wide variety of systems [See Schell and Ross, 1986, Inamdar *et al.*, 1991]. For example, input delays occur as one of the common sources of time delay, such as in chemical processes, transportation lags and measurement delays, etc. It is well known that the existence of time delay degrades the control performances and makes the closed-loop stabilization difficult. Hence, the study of time delay systems has been paid considerable attention

over the past years. The description of time-delay systems lead to differential-difference equations, which require the past values of the system variables. There are no known methods to get necessary and sufficient conditions for the robust stability of such systems. There are many approaches that yield sufficient conditions with varying degrees of sophistication.

In Chapter 4, a Variable Structure Control (VSC) design methodology is proposed to robustly stabilize an uncertain state-delay system with nonlinear and possibly mismatched uncertainties, utilizing the concept of perturbation compensation. The study of uncertain state-delay systems has received much attention by researchers in the past decade. One reason for this interest is that time delay is encountered in various engineering systems and can be the cause of instability. Another reason is that practical control systems unavoidably include uncertainty or disturbances due to modeling or measurement errors and other approximations. The sliding mode in VSC possesses well-known features that make it very attractive for control systems. These include fast response, insensitivity to parameter variations, and decoupling design procedure, among others. Sufficient robust stability conditions are derived which are independent of the size of the delay; furthermore no matching conditions on the uncertainty are assumed. The main advantages of the proposed approach are that *i)* a realizable control law is obtained, *ii)* chattering reduction/elimination is achieved, and *iii)* guidelines are given for the design of the control matrix involved in the specification of the switching function. Finally the results are illustrated with an open-loop unstable example.

Chapter 5 proposes an approach for uncertain input-delayed systems with linear and possibly mismatching uncertainties via Sliding Mode Control. Sufficient robust stability conditions are derived. We consider a transformation to convert the original system into an equivalent system without delay [Artstein, 1982]. This reduction enables the use of known stabilizing methods for linear systems without input delay. The main advantages of the proposed approach are that (i) a realizable control law is obtained, (ii) chattering

reduction is achieved since a continuous control law is used, (iii) guidelines are given for the design of the switching function to ensure the asymptotic stability of the normal system in the sliding mode. Certain open issues in the sliding mode literature are brought out and possible solutions are discussed.

Finally, future research directions and possible extensions to the presented work are discussed in chapter 6.

CHAPTER 2

THE NYQUIST ROBUST STABILITY MARGIN--A NEW METRIC FOR THE STABILITY OF UNCERTAIN SYSTEMS

2.1 Introduction

Many of the now well-known results on robust stability can trace their origins to the Nyquist stability criterion (SISO case), or the Generalized Nyquist stability criterion (MIMO case) [MacFarlane, 1980). Such is the case of the well known multivariable stability margin k_m [Safonov, 1982] and the structured singular value μ [Doyle, 1982]. Although the generalized Nyquist criterion addresses the issue of stability using complex variable (eigenvalue or transfer function) arguments, the predominant robustness analysis methods typically rely on arguments based on real singular value or magnitude bounds. One reason for this choice was clearly the more advantageous conditioning of singular values for numerical calculations. However, another important factor was the historical fact that uncertainty measurements were initially proposed for MIMO systems using singular value bounds, which collapse to magnitude (disk) bounds for SISO systems.

In this chapter we introduce the concept of the Nyquist robust-stability margin, k_N, for characterizing the closed-loop stability of uncertain systems. The approach makes direct use of Nyquist arguments and is based on the analysis of the perturbed eigenvalue loci, hence avoiding undue conservatism that may result through the use of singular-value upper bounds.

A key element in the new approach is *Critical Direction Theory* applied to uncertainties in the Nyquist plane. A basic version of the critical direction concept was first presented in [Guzzella, *et al.,* 1985] as a tool for analyzing the robust stability of polynomials with ellipsoidal uncertainties. In [Latchman and Crisalle, 1995] the critical

direction notion is redefined directly on the Nyquist plane to facilitate the derivation of necessary and sufficient robust stability conditions for transfer functions subject to arbitrary perturbations. In this paper we further extend the critical direction theory for the analysis of MIMO systems by formulating the concept in the generalized Nyquist plane, introducing new nomenclature to facilitate the seamless extension from the SISO to the MIMO case, and defining the new concept of the Nyquist robust stability margin.

The critical direction method is based on recognizing that, at any given frequency on the Nyquist plane, there is only one direction of perturbation of relevance to the stability analysis. This critical direction is defined by the oriented line that has its origin at the location of the nominal eigenvalue of the unperturbed system and passes through the critical point -1+j0. In fact, from a stability point of view all points on the uncertainty eigentemplate that do not lie on the critical line can be ignored. This allows the characterization of robust stability margins for uncertain systems characterized by irregular perturbation templates, a problem that poses significant challenges to other analysis methods. Other attractive features of the Nyquist robust-stability margin theory are that its MIMO form is a natural extension of the SISO case, and that its versions for continuous and discrete systems are formally identical.

The chapter is organized as follows. Section 2.2 gives a succinct review of robust-stability margin results for the SISO and MIMO cases. Section 2.3 formulates the critical direction theory for analyzing the robustness of uncertain SISO systems, and Section 2.4 gives the details of the new critical direction theory for the MIMO case. A SISO and a MIMO example are given in Section 2.5, followed by final remarks and conclusions in Section 2.6.

2.2 Frequency-Domain Approach to Robust Stability Analysis

The new Nyquist robust stability margin is an object derived using Nyquist arguments. In order to provide a contextual background for the ensuing discussions, in

this section we review relevant SISO and MIMO results on robust stability analysis, placing particular attention on the Nyquist arguments that also lie at the root of the new developments presented here.

The transfer function $g(s)$ of a SISO system can be written in terms of an additive perturbation $\delta(s)$ about a nominal transfer function $g_0(s)$

$$g(s) = g_0(s) + \delta(s), \ \delta(s) \in \boldsymbol{d} \tag{2.1}$$

where $\boldsymbol{d}$ represents the set of allowed perturbations. When all $\delta(s) \in \boldsymbol{d}$ are considered, then at each frequency the map $g(j\omega) = g_0(j\omega) + \delta(j\omega)$ defines a region denoted *uncertainty template* or value set [Barmish, 1993]. It is then possible to visualize the transfer-function uncertainty directly in terms of the classical Nyquist plot as uncertainty templates about the locus of the nominal system $g_0(j\omega)$.

The mathematical description for the set of perturbations may be formulated in different spaces. For example, the Laplace-function domain $\boldsymbol{d}$ is often defined implicitly in terms of a frequency-domain description. Such is the case where the uncertainty templates are specified to be circular at each frequency. This makes the analysis of robust stability particularly straightforward. On the other hand, the templates for a number of useful uncertainty descriptions are not circular but rather irregularly shaped, exhibiting highly directional features. This is the case for parametric uncertainty descriptions of the form $\delta(s) = g(s,\boldsymbol{p}) - g(s,\boldsymbol{p}_0)$, where $g(s,\boldsymbol{p}_0)$ and $\boldsymbol{p}_0$, respectively, denote the nominal transfer function and the nominal parameter set, and where $g(s,\boldsymbol{p})$ is characterized by m real, uncertain parameters $\boldsymbol{p} \in \boldsymbol{d}_p \subset \Re^m$. In this framework the set $\boldsymbol{d}$ is defined implicitly through the Euclidean domain $\boldsymbol{d}_p$. Typical uncertainty representations for the parameter uncertainty set $\boldsymbol{d}_p$ are polytopes and ellipsoids. The special case where the uncertainty templates are ellipses is an interesting example of directional uncertainty templates which also has a tractable mathematical representation.

The representation (2.1) is general and adequately encompasses additive perturbations as well as uncertainties that may appear as multiplicative perturbations of

the nominal system, as parametric variations on pole/zero locations, and/or variations of the coefficients of the numerator and denominator polynomials in the transfer function. The difference in each case is the shape and orientation of the uncertainty templates associated with $\delta(s)$.

Of particular interest are three types of SISO uncertainty descriptions. Specifically, we denote the set of uncertainties giving rise to *unstructured* (circular), *elliptical*, and *structured* (arbitrary) templates, respectively, by the notation $\boldsymbol{d_U}$, $\boldsymbol{d_E}$, and $\boldsymbol{d_S}$.

In the case of uncertain MIMO systems, the matrix transfer function $G(s)$can be written as an additive perturbation $\Delta(s)$ about a known nominal transfer matrix $G_o(s)$

$$\boldsymbol{G}(s) = \boldsymbol{G}_o(s) + \Delta(s), \ \Delta(s) \in D \tag{2.2}$$

where D is the set of allowed perturbations. The uncertainty set D is described in terms of available information about the modeling errors. We use the designation D_U to represent *Unstructured uncertainties* with a single norm bound, D_B to denote *block-diagonal norm-bounded uncertainties*, D_C to represent *element-wise circular (disk-bounded) uncertainties* and D_E to represent *element-by-element elliptical uncertainties*. Other uncertainty descriptions are defined in terms of errors in the elements of the state-space matrices, or errors in the real parameters of the matrix transfer functions. In all these cases the uncertainty description still can be mapped (via eigenvalue inclusion regions) on to frequency-response uncertainty templates, in the Nyquist plane which will then form the basis for defining the Nyquist robust stability margin.

Classical SISO Robust-Stability Results

The SISO robust stability analysis is concerned with the stability of the system that results when the system $g(s)$ in (2.1) is arranged in a unity negative-feedback configuration. It is normally assumed that *(i)* the nominal system is stable under unity

negative-feedback, and that (*ii*) the nominal and uncertain system have the same number of open-loop unstable poles. These assumptions are adopted throughout this paper.

The most studied case is perhaps that where the uncertainty description is unstructured, $\delta(s) \in \boldsymbol{d}_U$ *i.e.,* when $|\delta(j\omega)| \leq |W(j\omega)|$, where $W(s)$ is a known function whose frequency-response magnitude defines the radius of the circular uncertainty templates.

Invoking the Nyquist stability criterion leads to the necessary and sufficient stability requirement that all uncertainty templates exclude the critical point -1+j0, *i.e.*,

$$g(j\omega) = g_0(j\omega) + \delta(j\omega) \neq -1 \quad \forall \delta(j\omega) \text{ and } \forall \omega \tag{2.3}$$

This condition can be expressed through the inequality [Doyle *et al.*, 1992]

$$\left| \frac{W(j\omega)}{1 + g_0(j\omega)} \right| < 1 \quad \forall \omega \tag{2.4}$$

or equivalently,

$$\left| \frac{W(j\omega)}{1 + g_0(j\omega)} \right|_\infty < 1 \tag{2.5}$$

The robustness analysis for uncertainty descriptions with non circular templates is more challenging. For this reason it is quite common for elliptical or arbitrarily shaped uncertainty templates, arising for example from parametric uncertainties, to be circumscribed by an appropriate circle [Bhattachrya *et al.,* 1993]. Although this approach yields sufficient conditions for robust stability, it is nevertheless inherently conservative.

Classical MIMO Robust-Stability Results

As in the SISO case, the robustness analysis for MIMO systems makes use of the generalized Nyquist stability criterion [MacFarlane, 1980], where avoidance of the critical point 1+j0 is at the center of interest from the point of view of absolute stability assessment. This has been the major focus of much research interest in the development of robust multivariable stability margins. Consider the uncertain transfer matrix G(s) in

(2.2) with nominal model $G_o(s)$ affected by an uncertainty $\Delta(s)$. Again it is assumed that (*i*) the nominal system is stable under unity negative-feedback, and that (*ii*) the nominal and uncertain system have the same number of open-loop unstable poles.

The conditions under which no eigenvalue of the uncertain system $G_o(j\omega)+\Delta(j\omega)$ is equal to -1+j0, *i.e.*,

$$\lambda\big(G_o(j\omega)+\Delta(j\omega)\big)\neq -1 \quad \forall\Delta(j\omega) \text{ and } \forall\omega \tag{2.6}$$

is readily shown to lead to the determinantal condition

$$\det\big(\mathrm{I}+M(j\omega)\Delta(j\omega)\big)\neq 0 \quad \forall\Delta(j\omega) \text{ and } \forall\omega \tag{2.7}$$

where $M(j\omega):=\big(\mathrm{I}+G_o(j\omega)\big)^{-1}$.

Condition (2.7) is of course equivalent to the eigenvalue condition

$$\lambda\big(M(j\omega)\Delta(j\omega)\big)\neq -1 \quad \forall\Delta(j\omega) \text{ and } \forall\omega \tag{2.8}$$

The multivariable stability margin k_m and the structured singular value μ were independently defined by Safonov [1982] and Doyle [1982], respectively, in terms of the determinantal stability condition (2.7), rather than the eigenvalue conditions (2.6) or (2.8). For an uncertainty of a given description D, the multivariable stability margin k_m is defined as the matrix 2-norm of the smallest destabilizing uncertainty in the given class, namely:

$$k_m(\omega)=\min_{\Delta\in D}\Big\{\bar{\sigma}\big(\Delta(j\omega)\big):\ \det\big(\mathrm{I}+M(j\omega)\Delta(j\omega)\big)=0\Big\} \tag{2.9}$$

This definition corresponds to the reciprocal of the robustness measure μ defined in [Doyle, 1982], which is defined as the inverse of the norm of the smallest destabilizing uncertainty in the class, namely:

$$\mu(\omega)=\min_{\Delta\in D}\Big\{\bar{\sigma}\big(\Delta(j\omega)\big):\ \det\big(\mathrm{I}+M(j\omega)\Delta(j\omega)\big)=0\Big\}^{-1} \tag{2.10}$$

Note that if no uncertainty in the allowable class destabilizes the system (*i.e.*,, makes an eigenvalue of $G_o(j\omega)+\Delta(j\omega)$equal to -1 at any frequency) then $k_m = \infty$ and $\mu = 0$. These margins provide a measure of tolerable uncertainty size.

A number of powerful robust-stability results can be obtained in terms of the robust stability margins (2.9)and (2.10). Consider, for example, the following case of interest. An uncertainty description D is said to be *closed under contraction and rotation* if for any $\Delta(s) \in D$, then $\gamma e^{j\theta}\Delta(s) \in D$for all $0 \le \gamma \le 1$ and for all $0 \le \theta \le 2\pi$.. Starting from the eigenvalue stability condition (2.8), we can state the following lemma.

Lemma 2.1

Consider an uncertainty description D that is closed under contraction and rotation. Then the uncertain system $\boldsymbol{G}(s) = \boldsymbol{G}_o(s) + \boldsymbol{\Delta}(s)$ is stable for all $\boldsymbol{\Delta}(s) \in D$ if and only if

$$\sup_{\boldsymbol{\Delta}\in D} \rho\left(\boldsymbol{M}(j\omega)\boldsymbol{\Delta}(j\omega)\right) < 1 \quad \forall\omega \tag{2.11}$$

Proof: Since the eigenvalue condition(2.8) is necessary and sufficient for robust stability, it is clear that (2.11) immediately constitutes a sufficient stability condition. That (2.11) is also necessary is established by contraposition as follows: Consider an uncertainty $\boldsymbol{\Delta}_o(s) \in D$ such that $\rho\left(\boldsymbol{M}\boldsymbol{\Delta}_o\right) = [\gamma]^{-1} > 1$, then due to the closure under contraction we can always find an uncertainty $\boldsymbol{\Delta}_1 \in D$with $\boldsymbol{\Delta}_1 = \gamma\boldsymbol{\Delta}_o$; $0 < \gamma < 1$, such that $\rho\left(\boldsymbol{M}\boldsymbol{\Delta}_1\right) = 1$. Since the phase of the uncertainty class is arbitrary, due to the closure under rotation we can assign a scalar phase multiplier to $\boldsymbol{\Delta}_1$ to get $\boldsymbol{\Delta}_2$ such that $\lambda\left(\boldsymbol{M}\boldsymbol{\Delta}_2\right) = -1$, which of course implies instability. Thus, it follows that for the description D specified in the Lemma, condition (2.11) is both necessary and sufficient for robust stability. ∇

For the classes of uncertainties covered by Lemma 2.1, the spectral radius condition (2.11) is in fact equivalent (after appropriate normalizations) to the definitions

of μ and k_m and thus it naturally follows that μ is equal to the left hand side of (2.11). Hence the conclusion of Lemma 2.1 can be equivalently formulated as

$$k_m > 1 \quad \forall \omega \tag{2.12}$$

or

$$\mu < 1 \quad \forall \omega \tag{2.13}$$

The k_m and μ stability margins serve to provide valuable characterization of destabilizing uncertainties. Specific procedures and results have been developed for computing these stability margins (or at least good upper and lower bounds) for various useful classes of uncertainties. Additionally, the μ problem formulation has produced an extensive set of results associated with the class of block diagonal bounded uncertainties, and has thus attained wide acceptance for robustness analysis, and more recently, also for controller synthesis using the μ-synthesis method. In all these applications the computation of the stability margin is effected by calculating singular-value upper bounds which give sufficient, and in some cases necessary and sufficient, stability conditions.

Unfortunately, however, the use of norm-bounds, especially singular value bounds, in the characterization of destabilizing uncertainties often causes the rich structural properties, including phase and directionality to be ignored. On the other hand, as we show below significant advantages can be gained by studying the robustness problem directly from an eigenvalue point of view, especially since necessary and sufficient stability conditions correspond precisely to an eigenvalue condition via the generalized Nyquist criterion.

2.3 The Critical Direction Method for SISO Systems

In this section we present an exact robust stability result for the case of SISO systems using the critical direction theory. We adopt a nomenclature that permits the extension of the critical-direction concepts to the MIMO case. We also introduce the

definition of the Nyquist robust stability margin for SISO systems in a form which readily extends to the MIMO case.

Figure 2.1 shows a typical Nyquist diagram for a SISO system illustrating the *nominal frequency response* $g_o(j\omega)$ and an irregularly shaped uncertainty template. We define the *critical line* at a given frequency ω as the directed line which originates at the nominal point $g_o(j\omega)$ and passes through the critical point -1+j0. The figure is also useful for identifying the entities defined below:

1. *The critical direction*

$$d(j\omega) := -\frac{1+g_o(j\omega)}{|1+g_o(j\omega)|} \tag{2.14}$$

which may be interpreted as the unit vector that defines the direction of the critical line.

2. *The uncertainty template*

$$\mathcal{T}(\omega) := \left\{ g(j\omega) \,|\, g(j\omega) = g_o(j\omega) + \delta(j\omega),\ \delta(s) \in d \right\} \tag{2.15}$$

3. *The critical template*

$$\mathcal{T}_c(\omega) := \left\{ z \in \mathcal{T}(\omega) \,|\, z = g_o(j\omega) + \alpha d(j\omega) \ \textit{for some}\ \alpha \in \Re^+ \right\} \tag{2.16}$$

namely, the set of perturbed frequency-responses lying along the critical direction $d(j\omega)$.

4. *The critical perturbation radius*

$$\rho_c(\omega) := \max_{\alpha \in \Re^+} \left\{ \alpha \,|\, z = g_o(j\omega) + \alpha d(j\omega) \in \mathcal{T}_c(\omega) \right\} \tag{2.17}$$

As an illustration of the previous definitions note that the critical line at frequency $\omega = \omega_1$ is readily identified in Figure 2.1 as the directed line with origin at $g_o(j\omega_1)$ and passing through the point -1+j0. The critical direction $d(j\omega_1)$ is simply the unit-length vector that characterizes the direction of the critical line. In addition, the critical radius

$\rho_c(\omega_1)$ is shown as the distance between the nominal Nyquist point $g_o(j\omega_1)$ and the point where the critical line intersects with the boundary of the template. Finally, the critical template $\mathcal{T}_c(\omega)$ is also readily characterized in Figure 2.1 as the subset of the uncertainty template $\mathcal{T}(\omega)$ that intersects with the critical line.

By definition $\mathcal{T}_c(\omega)$ is a subset of the critical line, thus it follows that $\mathcal{T}_c(\omega)$ is either a single straight-line segment, or the union of such segments. The critical template may also contain isolated points should the boundary of the template $\mathcal{T}(\omega)$ be tangent to the critical direction. Figure 2.1 shows the case where the critical template is a continuous segment, and hence $\mathcal{T}_c(\omega)$ is a convex set even though the entire template is highly non-convex. The critical template is discontinuous when it is made up of the union of distinct segments. In this case $\mathcal{T}_c(\omega)$ is not convex, but each of its member segments is a convex subset.

With these definitions in hand we can proceed to state the following robust-stability theorem. For ease of exposition we assume that the critical template $\mathcal{T}_c(\omega)$ is a convex set for all frequencies. This restriction can be relaxed through obvious modifications to account for each of the convex segments of $\mathcal{T}_c(\omega)$.

Theorem 2.1

Let the nominal SISO system $g_o(s)$be subject to an uncertainty $\delta(s) \in \boldsymbol{d}$. Then the uncertain closed-loop system remains stable under unity feedback if and only if

$$\frac{\rho_c(\omega)}{\left|1+\lambda_c\left(g_o(j\omega)\right)\right|} < 1 \quad \forall \omega \tag{2.18}$$

The proof of the theorem is omitted since all details are given in [Latchman and Crisalle, 1995]. ∇

Motivated by (2.18) we now propose the following definition of the *Nyquist Robust-Stability Margin* for SISO systems:

$$k_N(\omega) := \frac{\rho_c(\omega)}{\left|1 + \lambda_c\left(g_o(j\omega)\right)\right|} \tag{2.19}$$

From Theorem 2.1 it follows that

$$k_N(\omega) < 1 \quad \forall \omega \tag{2.20}$$

is a necessary and sufficient condition for robust closed-loop stability. Furthermore, the quantity $\gamma = \left[k_N(\omega)\right]^{-1}$ specifies the amount by which the uncertainty template should be increased or decreased to attain the limiting case of stability.

The SISO critical direction result enables an exact assessment of stability for systems with directional uncertainty templates, without having to circumscribe the templates with larger circular uncertainties. The calculation of the critical perturbation radius $\rho_c(\omega)$ involves determining the intersection of a straight line (the critical direction) with a curve (the boundary of the template $\mathcal{T}(\omega)$). Section 2.5 shows an example where the critical radius can be calculated analytically.

It is also worth noting that the critical perturbation radius also gives a systematic methodology for determining the uncertainty weights which can be used as one of the inputs for various robust synthesis methods. Of particular interest in this regard are cases where there exists a *Critical Weighting Function*, $W_c(s)$, such that its magnitude satisfies the interpolation condition

$$\left|W_c(s)\right| = \rho_c(\omega) \quad \forall \omega \tag{2.21}$$

When (2.21) can be exactly satisfied, Theorem 2.1 leads to the familiar H_∞ condition

$$\left\| \frac{W_c(s)}{1 + g_o(s)} \right\|_\infty < 1 \tag{2.22}$$

which is of the form (2.4). If (2.22) cannot be satisfied exactly but can be approximated using standard frequency-domain regression methods, then the resulting approximate weight can still be used as the basis for practical robust-synthesis design for systems with highly structured templates.

It should be noted that Theorem 2.1 recovers as a special case the situation where the uncertainties are circular or disk-bounded. The interested reader is referred to [Latchman and Crisalle, 1995] for further details on this point.

In the following section we show how the critical direction approach and the Nyquist robust-stability margin may be generalized to the case of uncertain MIMO systems.

2.4 The Critical Direction Method for MIMO Systems

For MIMO systems we focus attention on the effect of uncertainties on the eigenvalue loci along the critical direction. This treatment has several desirable properties. First of all we derive necessary and sufficient robust stability conditions based upon eigenvalue relationships obtained directly from eigenvalue uncertainty templates. Thus, in principle this method yields a stability margin measure even in cases where singular value conditions fail to give necessary and sufficient stability conditions, provided that a method can be found to define tight inclusion regions for the uncertain eigenvalues. Furthermore by considering only the subset of the eigenvalue inclusion region that lies along the critical direction, we significantly reduce the computation involved in using the boundary of the eigentemplates (See E-contours [Daniel and Kouvaritakis, 1985]) for stability assessment.

Consider the generalized Nyquist plots in Figure 2.2 which (for illustrative purposes only) show two irregularly shaped eigenvalue inclusion regions (templates) about two of the n eigenvalues $\lambda_i(G_o(j\omega))$, i =1, 2,...,n. In analogy with the SISO case, for each of the n eigenloci we define at each frequency an associated critical line, defined as the directed line originating at the location of nominal eigenvalues $\lambda_i(G_o(j\omega))$ and passing through the critical point -1+j0. Also in analogy with the SISO case discussed in Section 2.3 we define the following entities:

1. The crtitical directions

$$d_i(j\omega) := -\frac{1+\lambda_i(\boldsymbol{G}_o(j\omega))}{\left|1+\lambda_i(\boldsymbol{G}_o(j\omega))\right|} \tag{2.23}$$

which may be interpreted as the unit vectors which define the direction of each critical line.

2. The uncertainty eigentemplates

$$\mathcal{T}_i(\omega) := \left\{ \lambda_i(\boldsymbol{G}(j\omega)) \,\middle|\, \lambda_i(\boldsymbol{G}(j\omega)) = \lambda_i(\boldsymbol{G}_o(j\omega) + \boldsymbol{\Delta}(j\omega)),\ \boldsymbol{\Delta}(s) \in D \right\} \tag{2.24}$$

3. The critical eigentemplates

$$\mathcal{T}_{ci}(\omega) := \left\{ z \in \mathcal{T}_i(\omega) \,\middle|\, z = \lambda_i(\boldsymbol{G}_o(j\omega)) + \alpha d_i(j\omega) \ \textit{for some}\ \alpha \in \Re^+ \right\} \tag{2.25}$$

4. The critical perturbation radii

$$\rho_{ci}(\omega) := \max_{\alpha \in \Re^+} \left\{ \alpha_i \,\middle|\, z = \lambda_i(\boldsymbol{G}_o(j\omega)) + \alpha_i d_i(j\omega) \in \mathcal{T}_{ci}(\omega) \right\} \tag{2.26}$$

The entities defined above have identical interpretations to their SISO counterparts defined in Section 2.2. Hence, their geometrical interpretation can be obtained directly from the SISO Nyquist diagram given in Figure 2.1, provided that the nominal frequency-response plot $g_o(j\omega)$ is substituted by an eigenvalue plot $\boldsymbol{G}_o(j\omega)$. Obviously, the tight E-contour templates for unstructured and structured uncertainty descriptions utilized in [Daniel and Kouvaritakis, 1985; Kouvaritakis and Latchman, 1985] are respectively equivalent to the uncertainty templates $\mathcal{T}_i(\omega)$.

Using these definitions we can now state the following robust stability theorem. As in the SISO case, for simplicity of exposition we assume that the critical uncertainty eigentemplates $\mathcal{T}_{ci}(\omega)$ are convex sets at all frequencies.

Theorem 2.2

Let the nominal MIMO system $G_o(s)$ be subject to an uncertainty $\Delta(s) \in D$. Then the uncertain closed-loop system remains stable under unity feedback if and only if

$$\max_{i=1,2,\cdots n} \frac{\rho_{c_i}(\omega)}{\left|1+\lambda_i\left(G_o(j\omega)\right)\right|} = \frac{\rho_c(\omega)}{\left|1+\lambda_c\left(G_o(j\omega)\right)\right|} < 1 \quad \forall \omega \tag{2.27}$$

where $\rho_c(.)$ and $\lambda_c(.)$ are respectively used to denote the critical perturbation radius and the eigenvalue associated with the eigenloci resulting from the maximization over all $i = 1,2,\cdots n$ in (2.27).

Proof. Assuming nominal closed loop stability and that the nominal and perturbed open loop systems have the same number of open loop unstable poles, the Generalized Nyquist stability criterion guarantees that the uncertain closed loop system is stable if and only if

$$\lambda_i\left(G_o(j\omega)+\Delta(j\omega)\right) \neq -1 \quad \forall i \text{ and } \forall \omega \tag{2.28}$$

Using the defining equation for the eigenvalues of $G_o(j\omega)+\Delta(j\omega)$, namely

$$\det\left(G_o(j\omega)-zI+\Delta(j\omega)\right)=0 \tag{2.28}$$

at each frequency ω we can parametrize the uncertain eigenvalues by

$$z = \lambda_i\left(G_o(j\omega)\right)+\rho_i e^{j\theta_i} \tag{2.29}$$

where $\theta_i = \theta_i(\omega)$ varies in the range $0 \le \theta_i \le 2\pi$, and for each value of θ_i the scalar $\rho_i = \rho_i(\omega)$ varies in the range $0 \le \rho_i \le \overline{\rho}_i$ where $\lambda_i\left(G_o(j\omega)\right)+\overline{\rho}_i e^{j\theta_i}$ is an eigenvalue corresponding to the boundary of the eigentemplate $T_i(\omega)$. Then condition (2.28) can be written as

$$1+\lambda_i\left(G_o(j\omega)\right)+\overline{\rho}_i e^{j\theta_i} \neq 0 \quad \forall i \tag{2.30}$$

Since the term $1+\lambda_i\left(G_o(j\omega)\right)$ is fixed, the only possibility for violating the stability condition (2.30) is for $\rho_i e^{j\theta_i}$ to be oriented along the critical direction, namely

$$\rho_i e^{j\theta_i} = \alpha_i(\omega) d_i(j\omega) \tag{2.31}$$

where $0 \le \alpha_i(\omega) \le \bar{\rho}_{c_i}$. Thus, for robust stability we have the necessary and sufficient stability condition

$$1 + \lambda_i\big(\boldsymbol{G}_o(j\omega)\big) + \alpha_i(\omega) d_i(j\omega) \neq 0 \quad \forall i \ \textit{and} \ \forall \omega \tag{2.32}$$

Invoking definition (2.23) it follows that a sufficient stability condition is given by

$$\frac{\alpha_i(\omega)}{\left|1 + \lambda_i\big(\boldsymbol{G}_o(j\omega)\big)\right|} < 1, \ 0 \le \alpha_i(\omega) \le \rho_{c_i}, \ \forall i \ \textit{and} \ \forall \omega \tag{2.33a}$$

Furthermore, since along the critical direction $\alpha_i(\omega) \le \rho_{c_i}(\omega)$ stability is ensured if

$$\frac{\rho_{c_i}(\omega)}{\left|1 + \lambda_i\big(\boldsymbol{G}_o(j\omega)\big)\right|} < 1 \quad \forall i \ \textit{and} \ \forall \omega \tag{2.33b}$$

The proof is completed by taking the maximum over all i and noting that, because of the convexity assumption on the critical templates $\mathcal{T}_{c_i}(\omega)$ conditions (2.33a)-(2.33b) are also *necessary* for stability. ∇

In complete parallel with the SISO case

$$k_N(\omega) := \frac{\rho_{c_i}(\omega)}{\left|1 + \lambda_i\big(\boldsymbol{G}_o(j\omega)\big)\right|} \tag{2.34}$$

defines the MIMO *Nyquist Robust-Stability Margin*, and the scalar $\gamma = \big[k_N(\omega)\big]^{-1}$ again specifies the amount by which the uncertainty eigentemplate should be increased or decreased to attain the limiting case of stability. Consequently, from Theorem 2.2 it follows that a necessary and sufficient condition for robust stability is given by the Nyquist-derived constraint

$$k_N(\omega) < 1 \quad \forall \omega \tag{2.35}$$

The Nyquist robust stability condition (2.35) provides an exact answer to the robust stability margin problem, much in the same vein as the margins $k_m(\omega)$ and $\mu(\omega)$.

A most pleasing feature of the new stability measure is that the SISO version (2.20) and the MIMO version (2.35) of the Nyquist robust stability margin are formally identical. Much more importantly, the $k_N(\omega)$ formalism adheres to the classical generalization of SISO system properties to the MIMO case via the vehicle of the eigenvalues of the frequency response matrix.

Relationship between $k_m(\omega), \mu(\omega)$, and $k_N(\omega)$

In light of Theorem 2.2 and the remarks above, it is clear that there are strong equivalences between the Nyquist stability margin $k_N(\omega)$ and the margins $k_m(\omega)$) and $\mu(\omega)$. To see this recall that the definition (2.9) for $k_m(\omega)$ is equivalent to finding at each frequency the smallest destabilizing Δ such that

$$1+\lambda_i\big(M(j\omega)\Delta(j\omega)\big) \neq 0 \quad \forall i \tag{2.36}$$

which is precisely the condition exploited in Theorem 2.2 to get the necessary and sufficient stability condition

$$1+\lambda_i\big(G_o(j\omega)\big)+\alpha_i(\omega)d_i(j\omega)\neq 0 \quad \forall i \ \ and \ \ \forall \omega \tag{2.37}$$

For uncertainties which satisfy the condition of Lemma 2.1, from (2.12)-(2.13) it follows that

$$k_N(\omega)<1 \Leftrightarrow \mu(\omega)<1 \Leftrightarrow k_m(\omega)>1 \Leftrightarrow \sup_{\Delta\in D}\rho(M\Delta)<1 \tag{2.38}$$

In fact, even for uncertainties for which Lemma 2.1 does not hold, it is still true that for all $\Delta(s)\in D$

$$1+\lambda_i\big(M(j\omega)\Delta(j\omega)\big) \neq 0 \Leftrightarrow k_N(\omega)<1 \tag{2.39}$$

The practical utility of Theorems 2.1 and 2.2 requires the computation of the critical perturbation radius $\rho_c(\omega)$. Several methods are presently being developed to exploit the critical direction theory to compute $\rho_c(\omega)$ and hence $k_N(\omega)$ for the case of affine and multi-affine parametric uncertainties in SISO and MIMO systems. The results

thus far are very encouraging, and have shown significant computational and algorithmic simplifications due to the focus on the critical direction. For the case of structured and unstructured MIMO uncertainties, tight eigenvalue inclusion regions may be obtained using the singular-value based E-contours method [Daniel and Kouvaritakis, 1993; Kouvaritakis and Latchman, 1985], with similarity or non-similarity scaling deployed to remove or reduce conservatism.

2.5 Examples

SISO Example

This section presents an example of a SISO system with an uncertainty description motivated from statistical parameter-estimation techniques. The uncertainty considered is an ellipsoidal parameter-space model which, in addition to its intrinsic merits, facilitates the direct analysis in the Nyquist plane permitting an explicit characterization of the Nyquist robust-stability margin $k_N(\omega)$,thus illustrating the application of the critical direction theory proposed in this paper. Furthermore, a discrete-time representation of the model is deliberately chosen to emphasize the fact that the critical direction method and its attendant Nyquist robust-stability margin concept are applicable to both the continuous and discrete domains.

Let us consider a general SISO system model given by the discrete-time transfer function

$$H(z;p) = \sum_{k=1}^{q} h_k z^{-k} \tag{2.40}$$

with q parameters defined by $\boldsymbol{h} = \left[h_1, h_2, \cdots h_q\right]^T$. The nominal system, denoted $H_o(z)$, is obtained when $\boldsymbol{h}$ assumes the nominal values $\boldsymbol{h}_o = \left[h_1^0, h_2^0, \cdots h_q^0\right]^T$, so that the real system is modeled as $H(z) = H_o(z) + \delta H(z)$. Let $\boldsymbol{h}$ in $\Re^q$ be such that $\boldsymbol{h} = \boldsymbol{h}_o + \delta\boldsymbol{h}$. Then we define the uncertain ellipsoidal parametric uncertainty description

$$\delta h = h - h_o \in D_h \tag{2.41}$$

$$D_h = \left\{ \delta h \in \Re^q \text{ s.t } \delta h^T Q_h^{-1} \delta h;\ Q_h = Q_h^T > 0 \right\} \tag{2.42}$$

We argue that, in addition to the remarkable mathematical tractability which we shall show later, the ellipsoidal parametric description of uncertainties is quite natural in many applications and offers, in contrast to hyper-rectangular descriptions, the further advantage of allowing the dependence among various system parameters to be taken into explicit account. Ellipsoidal models often arise quite naturally, as for example, whenever linear regression or least-squares analysis is used in model estimation [Guzzella, *et al.,* 1991; Kosut *et al.,* 1992].

Clearly the nominal parameter vector h_o defines the nominal transfer function $H_o(z) = \sum_{k=1}^{q} h_k^o z^{-k}$. Consider now the frequency response $H(e^{j\omega})$ where $H(z) = \sum_{k=1}^{q} h_k z^{-k}$ contains parameters $h_o = \left[h_1^0, h_2^0, \cdots h_q^0\right]^T$ belonging to the parameter ellipsoid. Under these conditions the following lemma shows that the parameter space ellipsoid maps precisely to ellipses at all frequencies except for $\omega = 0$ and $\omega = \pi$. Let Re (z) and Im (z) represent respectively the real and imaginary components of a complex number z.

Lemma 2.2

The parameter space ellipsoid defined by (2.42) maps to the elliptical uncertainty template

$$\Upsilon(\omega) = \left\{ z(\omega) = x_1(\omega) \,|\, x_2(\omega) = 0, \left| x_1(\omega) - x_1^0(\omega) \right| \le \sqrt{v_R(\omega)^T Q_h v_R(\omega)}, \omega = 0 \quad \omega = \pi \right\} \tag{2.43}$$

where

$$X(\omega) := \left[x_1(\omega)\ \ x_2(\omega)\right]^T \in \Re^2, \quad X_o(\omega) := \left[x_1^0(\omega)\ \ x_2^0(\omega)\right]^T \in \Re^2$$

such that

$$X(\omega) = \begin{bmatrix} \text{Re } \mathrm{H}\left(e^{j\omega}\right) \\ \text{Im } \mathrm{H}\left(e^{j\omega}\right) \end{bmatrix} = V(\omega)\boldsymbol{h}$$

$$X_o(\omega) = \begin{bmatrix} \text{Re } \mathrm{H}_o\left(e^{j\omega}\right) \\ \text{Im } \mathrm{H}_o\left(e^{j\omega}\right) \end{bmatrix} = V(\omega)\boldsymbol{h}_o$$

$$Q_\omega \;:=\; V(\omega)Q_h V(\omega)^T \in \Re^{2x2}$$

$$V(\omega) \;:=\; \begin{bmatrix} \cos\omega & \cos 2\omega & \cdots & \cos q\omega \\ \sin\omega & \sin 2\omega & \cdots & \sin q\omega \end{bmatrix}$$

At $\omega = 0$ *and* $\omega = \pi$, *matrix* $V(\omega)$ *is singular and the uncertainty template is entirely real and given by*

$$\mathcal{T}(\omega) = \left\{ z(\omega) = x_1(\omega) \mid x_2(\omega) = 0, \left| x_1(\omega) - x_1^0(\omega) \right| \le \sqrt{v_R(\omega)^T Q_h v_R(\omega)}, \omega = 0 \quad \omega = \pi \right\}$$

where $v_R(\omega) := \begin{bmatrix} \cos\omega & \cos 2\omega & \cdots & \cos q\omega \end{bmatrix}^T$

Proof. The proof makes use of a result given in [Guzzella, *et al.*, 1985]. Details are omitted. ∇

Our intention is to characterize the Nyquist robust-stability margin for the unity feedback system comprised of the nominal discrete-time system and its associated parametric uncertainty ellipsoid (2.42). Note that the robust stability of the closed-loop system cannot be analyzed using the l^2 results in [Tsypkin and Polyak, 1991] because their method is applicable only to continuous-time systems and to ellipsoidal uncertainty descriptions where the matrix Q_h is diagonal (*i.e.*, the principal directions are aligned with the coordinate axes). In fact, none of the conventional methods for analyzing robust stability of SISO systems appears capable of treating in a systematic fashion the case of ellipsoidal uncertainties D_h considered in this example.

Theorem 2.3

Under the assumptions of nominal closed-loop stability and that the nominal and perturbed systems share the same number of open-loop unstable poles, the unity negative feedback system with open-loop transfer function (2.40) and uncertain parameters (2.42) is stable for all $\delta h \in D_h$ *if and only if*

$$k_N(\omega) < 1 \ \forall \ \omega \tag{2.44}$$

where

$$k_N(\omega) = \frac{1}{\sqrt{d_c^T(\omega) Q_\omega^{-1} d_c(\omega)}} \quad \text{for } \omega \in (0,\pi) \tag{2.45}$$

$$k_N(\omega) = \sqrt{\frac{v_R(\omega)^T Q_h v_R(\omega)}{d_c^T(\omega) d_c(\omega)}} \quad \text{for } \omega = (0,\pi) \tag{2.46}$$

and

$$d_c(\omega) := \begin{bmatrix} -1 \\ 0 \end{bmatrix} - V(\omega) h_o \tag{2.47}$$

Proof. Inequality (2.44) is a direct result of applying the necessary and sufficient condition (2.18) of Theorem 2.1 in the Nyquist robustness margin form (2.20). It then suffices to prove (2.45) and (2.46). First consider the case where $\omega \in (0,\pi)$. Using two-dimensional vector analysis and the definitions given in Section 3, it is clear that the components of $d_c(\omega)$ given by (2.47) are, respectively, the unnormalized real and imaginary parts of the critical direction $d(j\omega)$ as defined in (4.1). Now from Lemma 2, at each ω the frequency-response $H\left(e^{j\omega}\right)$ is located inside an elliptical template with center at the point $H_o\left(e^{j\omega}\right)$. Clearly, the elliptical template is a convex set because it is continuous along any ray. A fundamental result from two-dimensional coordinate geometry gives the length of the line joining the center of an ellipse to the point of intersection along the vector $d_c(\omega)$ as

$$\rho_c(\omega) = \sqrt{\frac{d_c^T(\omega) d_c(\omega)}{d_c^T(\omega) Q_\omega^{-1} d_c(\omega)}} \tag{2.48}$$

Recognizing that

$$\left|1 + H_o\left(e^{j\omega}\right)\right| = \sqrt{d_c^T(\omega) d_c(\omega)} \tag{2.49}$$

it readily follows that

$$k_N(\omega) = \frac{\rho_c(\omega)}{\left|1 + H_o\left(e^{j\omega}\right)\right|} \tag{2.50}$$

is of the form (2.45). Analogously, the proof for the singular cases $\omega = 0, \pi$ is derived noting that from Lemma 2.2 it follows that at these frequencies $\rho_c(\omega) = \sqrt{v_R(\omega)^T Q_h v_R(\omega)}$ from which (2.46) is readily established.

In summary, for the case of ellipsoidal parametric uncertainties, the critical direction theory permits the exact characterization of the Nyquist robust-stability margin, and the derivation of the exact necessary and sufficient condition (2.44) for robust stability. ∇

MIMO Example

Consider the MIMO nominal transfer function

$$G_o(s) = 0.5 \begin{bmatrix} \dfrac{30}{(s+1)(s+2)(s+3)} & \dfrac{-3}{(s+4)} \\ \dfrac{7}{(s+5)} & \dfrac{10}{(s+1)} \end{bmatrix}$$

and the associated element-by-element elliptical uncertainty description.

$$D_E = \left\{\Delta(s) \text{ s.t. } \Delta_{ik}(j\omega) \in E_{ik}(j\omega) \ \forall \omega\right\} \tag{2.51}$$

where the boundary $\partial E_{ik}(j\omega)$ of each elliptical domain $E_{ik}(j\omega)$ is given by the map

$$\partial E_{ik}(j\omega) = 2A_{ik} e^{j\phi_{ik}} \cos\theta_{ik} + j\ 2B_{ik} e^{j\phi_{ik}} \sin\theta_{ik}, \ i,k = 1,2 \tag{2.52}$$

and where the coefficients A_{ik} and B_{ik} are the elements of the frequency-dependent matrices $A(\omega)$ and $B(\omega)$, and the major-axis orientation parameters ϕ_{ik} are the elements of the matrix $\phi(\omega)$. The objective is to illustrate the calculation of the Nyquist robust stability margin $k_N(\omega)$ at the frequency $\omega_1 = 1.21$, where

$$A(\omega_1) = \begin{bmatrix} 0.1264 & 0.0359 \\ 0.0680 & 0.3185 \end{bmatrix}, \quad B(\omega_1) = \begin{bmatrix} 0.0246 & 0.0278 \\ 0.0537 & 0.2707 \end{bmatrix}, \quad \phi(\omega_1) = \begin{bmatrix} 2.1799 & 4.5759 \\ 1.4906 & 3.8865 \end{bmatrix}$$

Note that formulations based on the multivariable stability margin $k_m(\omega)$ or the structured singular value $\mu(\omega)$ cannot currently offer an obvious approach for calculating the stability margins for the uncertainty description D_E.

At each frequency of interest we propose the following calculation procedure. First, the nominal eigenvalues $\lambda_i(G_o(j\omega))$ and the critical directions $d_i(\omega), i = 1,2$ are computed.

Second, for each eigenvalue, z satisfying $z_i = \lambda_i + \rho_i d_i$ is found. In the above expression, D is a diagonal similarity scaling matrix, P is the diagonal matrix where each diagonal element is the radius of the circle circumscribing the elliptical uncertainties. E_1 and E_2 are derived from the diagonalization of the uncertainty [Kouvaritakis and Latchman, 1985] and λ_i represents the eigenvalues of G_o. This yields the upper bound for $\lambda_i(G_o + \Delta)$ in the critical direction. Then an optimization is carried out over all θ_{ik} minimizing the function

$$f = \left| z_i^o - \lambda_i(G_o + \Delta) \right|^2 + \left| d_i - \frac{1 + \lambda_i(G_o + \Delta)}{\left| 1 + \lambda_i(G_o + \Delta) \right|} \right|^2 \tag{2.53}$$

Equation (2.53) gives the eigenvalue of $(G_o + \Delta)$ in the critical direction, which is closest to the upper bound z_i^o. The basic principle is similar to that described in [Kouvaritakis *et al.*, 1991].

At the selected frequency $\omega_1 = 1.21$, the nominal eigenvalues for the system considered in this example are $\lambda_1(G_o(j\omega_1)) = -0.2005 - j1.2313$, and

$\lambda_2(G_o(j\omega_1)) = 1.9332 - j2.4525$ and have the associated critical directions $d_1(j\omega_1) = -0.5446 + j0.8387$ and $d_2(j\omega_1) = -0.7672 + j0.6414$. Then using the structured E-contour method for the fixed critical directions $d_1(j\omega_1)$ and $d_2(j\omega_1)$ yields the critical radii $\rho_{c_1}(\omega_1) = 0.3140$ and $\rho_{c_2}(\omega_1) = 0.6445$. Finally, from (2.27) and (2.34) it is found that

$$k_N^{D_E}(\omega) = \max\left\{ \frac{\rho_{c1}(\omega_1)}{|1+\lambda_1(G_o(j\omega_1))|}, \frac{\rho_{c2}(\omega_1)}{|1+\lambda_2(G_o(j\omega_1))|} \right\} = \max\{0.2139, 0.1686\} \quad (2.54)$$

Hence, the Nyquist robust stability margin is $k_N^{D_E}(\omega_1) < 1$, and it is concluded that the system satisfies the necessary and sufficient condition for robust stability at this particular frequency.

Figure 2.3 shows a Nyquist diagram with the eigen-plots for the nominal system considered in this example. The eigentemplates $\mathcal{T}_1(\omega_1)$ and $\mathcal{T}_2(\omega_1)$ are not shown in the figure because for the purpose of determining the Nyquist robust-stability margin it is not necessary to calculate the entire templates. However, for reference the figure shows the Optimal D-scaling bounds obtained using the scalings given in [Kouvaritakis and Latchman, 1985]. The actual eigentemplates are bounded by these Optimal D-scaling contours. The points denoted $z_1(j\omega_1)$ and $z_2(j\omega_1)$ are on the boundary of the critical templates $\mathcal{T}_{c_1}(\omega_1)$ and $\mathcal{T}_{c_2}(\omega_1)$, respectively, and were obtained using the E-contour method. Note that for this example the Optimal D-scaling bounds provide a good estimate for the boundary point $z_1(j\omega_1)$, but yield a poor estimate for $z_2(j\omega_1)$. Note also that the critical eigentemplates $\mathcal{T}_{c_i}(\omega_1)$ and $\mathcal{T}_{c_2}(\omega_1)$ can be readily identified from the figure as the straight-line segments joining each nominal eigenvalue $\lambda_i(G_o(j\omega_1))$ with its corresponding boundary point $z_i(j\omega_1)$.

2.6 Conclusions

In this paper we have proposed the Nyquist Stability Margin, $k_N(\omega)$ as a new metric for robustness analysis of SISO and MIMO systems. The definition of new

stability margin is based on the Critical Direction Theory which provides a single framework for robustness analysis for SISO and MIMO systems. The analysis methodology makes direct use of the generalized Nyquist diagram, and in contrast to the prevalent approaches which emphasize singular-value perturbations, it focuses attention on eigenvalue perturbations.

The main advantage of the critical-direction theory is that it provides necessary and sufficient conditions for robust stability in the presence of highly structured uncertainties with phase and directionality constraints. Other approaches to these problems either do not have the inherent capability to deal with these structural details, or the directionality and phase constraints are deliberately ignored, giving rise in either case to sufficient-only conditions such as those associated with singular-value theory. On the other hand the new method explicitly exploits the detailed directionality and phase constraints of the uncertainties as these are manifested in the frequency domain uncertainty templates. Thus, the new method is applicable to a number of uncertainty descriptions for which other methods fail, such as the case of element-by-element ellipsoidal uncertainties in the transfer-function matrix, and other uncertainty descriptions with highly directional frequency-domain templates.

The new critical-direction technique opens up new avenues for robustness analysis and could lead to novel approaches for robust control synthesis. There is significant promise for fruitful new results in this area where the computational efforts are concentrated on a single and well-defined frequency-dependent directed line.

Nomenclature for Chapter 2

s	Laplace variable
$d_c(j\omega)$	direction in the complex plane
$g_o(j\omega)$	Nominal SISO transfer function
$g(j\omega)$	Uncertain SISO transfer function
$\boldsymbol{G}_o(j\omega)$	Nominal MIMO transfer function
$\boldsymbol{G}(j\omega)$	Uncertain MIMO transfer function
$k_N(\omega)$	Nyquist robust stability margin
$k_m(\omega)$	Multivariable robust stability margin
$\boldsymbol{M}(j\omega)$	Augmented system matrix
$W_c(s)$	SISO weight

Greek Letters

ω	Frequency
$\lambda_i(\boldsymbol{G}(j\omega))$	'i'th eigen value
$\boldsymbol{\Delta}(s)$	Uncertainty description
$\rho_c(\omega)$	Critical perturbation radius
$\mu(\omega)$	Structured singular value
$\overline{\sigma}(.)$	Maximum singular value
$\rho(.)$	Spectral radius
$\mathcal{T}(\omega)$	Eigen template
$\mathcal{T}_c(\omega)$	Critical eigen template

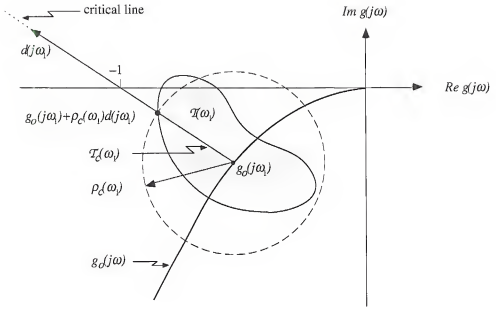

Figure 2.1. An irregularly shaped uncertainty template at frequency ω_1 (shaded area) and its critical perturbation radius $\rho_c(\omega_1)$. The critical template $T_c(\omega_1)$ (solid line) is the subset of template points lying on the line segment with end points $g_o(\omega_1)$ and $g_o(\omega_1)+\rho_c(\omega_1)d(j\omega_1)$.

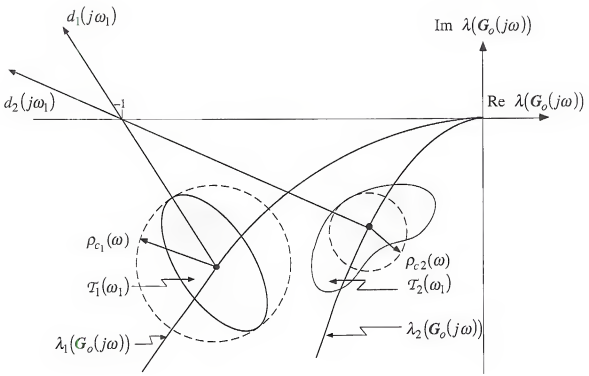

Figure 2.2. Nyquist plot of the eigenvalues of a 2x2 MIMO system showing two irregularly shaped eigen templates $\mathcal{T}_1(\omega_1)$ and $\mathcal{T}_2(\omega_1)$ (shaded areas) at frequency $\omega = \omega_1$.

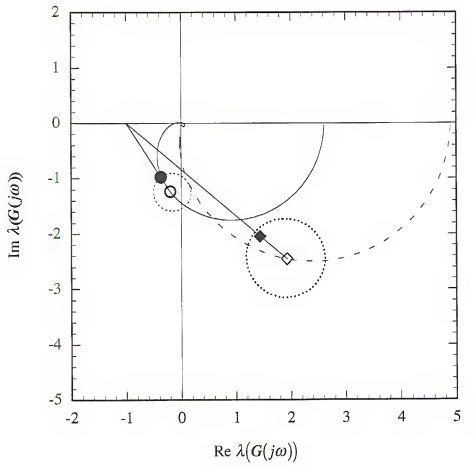

Figure 2.3. Nyquist plot of the MIMO Example. The contours represent the optimal D-scaling bounds for the eigentemplates $\mathcal{T}_1(\omega_1)$ and $\mathcal{T}_2(\omega_1)$ at the frequency $\omega_1 = 1.21$. The points $z_1(\omega_1)$ and $z_2(\omega_1)$ lie on the intersection of the boundary of their respective eigentemplates with the critical direction

CHAPTER 3

A NEW PERSPECTIVE ON COMPUTING ROBUST STABILITY MARGINS FOR COMPLEX PARAMETRIC UNCERTAINTIES

3.1 Introduction

Considerable attention has been given to the problem of assessing robust stability and computing stability margins for SISO systems with parametric uncertainties. It is now well known that affine parametric uncertainty problems result in a rank-one μ formulation. While the general robust stability margin computation problem is known to be NP-hard [Braatz *et al.*, 1994], its rank-one counterpart results in a convex optimization problem that can be readily computed, and can be often derived as an explicit analytical expression. This aspect of the problem is amply documented in the literature [Qiu and Davison, 1989]-[Hinrichsen and Pritchard, 1992] [Chen *et al.*, 1994a]-[Chen *et al.*, 1994b]. Of particular interest are references [Chen *et al.*, 1994b] and [Young, 1994], which establish connections between frequency-based stability conditions and polynomial conditions in the spirit of Kharitonov's theorem [Kharitonov, 1979].

This chapter focuses on the complex parametric uncertainty problem from the perspective of the recently proposed *critical direction* paradigm [Latchman and Crisalle, 1995] [Latchman *et al.*, 1997]. The approach followed in critical direction theory consists of first mapping the parameter space uncertainties into the Nyquist plane in the form of value sets or uncertainty templates, and then invoking specific directionality and phase properties to infer stability. The method is based on recognizing that at any given frequency on the Nyquist plane, the only uncertainties of relevance to the stability analysis are those which lie on a well defined *critical direction*. In fact, from a stability perspective all points on the uncertainty template that do not lie in the critical direction

can be ignored. This approach makes it possible to solve problems with highly directional uncertainties, such as SISO elliptical uncertainties, for which, at this time, approaches based on the more traditional structured singular value methods are not obvious.

This chapter addresses the case of element-wise complex parametric uncertainties where each individual uncertainty lies in a highly structured domain that is convex along any line from the origin. This is a mild convexity condition, since the actual domain could be highly non-convex. Exact analytical results are derived for geometrically simple uncertainty sets such as ellipses and rectangles. For the special case of disk-bounded affine uncertainties, the approach recovers in a transparent manner relevant results by previous methods [Hinrichsen and Pritchard, 1992][Chen *et al.,* 1995b].

Section 3.2 of the chapter provides background information and mathematical preliminaries, presents the uncertainty descriptions considered, and gives a brief review of the main elements of the critical direction theory. The main results are presented in Section 3.3, and connections with the classical approach based on M-D structures is explored in Section 3.4. Concluding comments are given in Section 3.5.

3.2 Background and Preliminaries

Consider a SISO system with an open-loop transfer function

$$g(j\omega,\boldsymbol{q}) = \frac{n(j\omega,\boldsymbol{q})}{d(j\omega,\boldsymbol{q})} \tag{3.1}$$

where $n(j\omega,\boldsymbol{q})$ and $d(j\omega,\boldsymbol{q})$ are complex polynomials that depend on an uncertainty vector

$$\boldsymbol{q} = \left[q_1(j\omega), q_2(j\omega), \cdots, q_m(j\omega)\right]^{\mathrm{T}} \in Q \subset C^m \tag{3.2a}$$

whose components $q_k(j\omega)$ belong to complex domains Q_k, *i.e.,*

$$q_k(j\omega) \in Q_k \subset C, \qquad k = 1, 2, \cdots, m \tag{3.2b}$$

and hence the uncertainty domain is the cartesian product space $Q = Q_1 \times Q_2 \times \cdots \times Q_m$. Figure 3.1 illustrates a case of a radially convex domain Q_k. Note that the entire domain is non-convex.

The uncertainty class considered in this s are described by element-wise uncertainty sets Q_k, k = 1, 2, ..., m, that are closed under contraction (i.e., are radially convex), and are unconstrained in phase. Letting ∂Q_k denote the boundary of domain Q_k, then the boundary elements $\partial q_k(j\omega) \in \partial Q_k$ are of the form

$$\partial q_k = \gamma_k(\omega, \theta_k) e^{j\theta_k}, \qquad k = 1, 2, \cdots, m \tag{3.2c}$$

where $0 \le \theta_k \le 2\pi$ is an unconstrained phase angle, and $\gamma_k(\omega, \theta_k) = |q_k(j\omega)| \ge 0$ is the frequency- and phase-dependent magnitude of the boundary element.

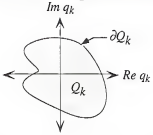

Figure 3.1 Radially convex uncertainty domain Q_k for complex component q_k

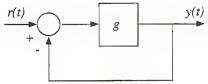

Figure 3.2 Unity negative feedback configuration.

The elements of the uncertainty vector $\boldsymbol{q}$ are assumed to appear in an affine fashion in (3.1), *i.e.*, the uncertain model is of the form

$$g(j\omega, \boldsymbol{q}) = \frac{n(j\omega, \boldsymbol{0}) + \sum_{k=1}^{m} n_k(j\omega) q_k(j\omega)}{d(j\omega, \boldsymbol{0}) + \sum_{k=1}^{m} d_k(j\omega) q_k(j\omega)} \tag{3.3}$$

Under the unity negative-feedback arrangement shown in Figure 3.2, the characteristic polynomial for system (3.3) is given by the polynomial sum

$$p(j\omega, \boldsymbol{q}) = n(j\omega, \boldsymbol{q}) + d(j\omega, \boldsymbol{q}) \tag{3.4}$$

It follows that the robust stability of the closed loop with respect to the uncertainty set Q can be evaluated by studying the uncertain polynomial (3.4). It is convenient to decompose (3.4) into its nominal and uncertain parts $p(j\omega,\boldsymbol{q}) = p_o(j\omega) + \delta(j\omega,\boldsymbol{q})$, where

$$p_o(j\omega) = n(j\omega,0) + d(j\omega,0) \tag{3.5a}$$

and

$$\delta(j\omega,\boldsymbol{q}) := \sum_{k=1}^{m} p_k(j\omega) q_k(j\omega) \tag{3.5b}$$

where

$$p_k(j\omega) := n_k(j\omega) + d_k(j\omega) = |p_k(j\omega)| e^{j\theta_{p_k}(j\omega)} \tag{3.6}$$

The variable $\theta_{p_k}(j\omega)$ is simply the frequency-dependent phase of polynomial $p_k(j\omega)$. Hence,

$$p(j\omega,\boldsymbol{q}) = p_o(j\omega) + \sum_{k=1}^{m} p_k(j\omega) q_k(j\omega) \tag{3.7}$$

Definition. *The Parametric Robust Stability Margin is the entity defined by the minimization expression*

$$\alpha^*(\omega) = \min_{\alpha \in \Re^+} \{\alpha \mid 0 \in p(j\omega,\alpha\boldsymbol{q}) \quad \text{for some } q \in Q\} \tag{3.8}$$

The parametric robust stability margin is a non-negative real scalar that can be interpreted as the minimal magnification $\alpha^*(\omega) > 1$ or contraction $\alpha^*(\omega) < 1$ of the uncertainty set Q that brings the closed-loop system to the edge of stability. Geometrically, the parametric robust stability margin represents the minimum tolerable blow-up factor. Note that the parametric robust stability margin is defined for each frequency.

Specific Parameter Uncertainty Descriptions Considered

Three types of uncertainty descriptions of the form (3.2c) are considered, namely the *(i)* circular, *(ii)* elliptical, and *(iii)* rectangular element-wise complex domains illustrated in Figure 3.3.

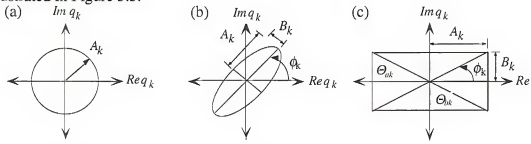

Figure 3.3 Three types of element-wise complex parametric uncertainty regions with unconstrained phase: (a) circular, (b) elliptical, and (c) rectangular domains.

Circular Uncertainties

As depicted in Figure 3.3a, this is the familiar case of circular (disk-bounded) uncertainties of radius $A_k(\omega)$ where

$$\gamma_k(\omega,\theta_k) = A_k(\omega) \;\; \forall \theta_k \tag{3.9}$$

Elliptical Uncertainties

Let $A_k(\omega) > 0$, $B_k(\omega) > 0$, and $\phi_k(\omega)$ respectively represent the frequency-dependent semi-major axis, semi-minor axis, and the orientation with respect to the real axis of the an ellipse centered at the origin of the complex plane as shown in Figure 3.3b. Then the boundary of each elliptical domain is given by the map

$$\partial q_k(\omega) = A_k(\omega)e^{j\phi_k(\omega)}\cos\big(\theta_k - \phi_k(\omega)\big) + jB_k(\omega)e^{j\phi_k(\omega)}\sin\big(\theta_k - \phi_k(\omega)\big)$$

from which it follows that

$$\gamma_k(\omega,\theta_k) = \Big[A_k(\omega)^2\cos^2\big(\theta_k - \phi_k(\omega)\big) + B_k(\omega)^2\sin^2\big(\theta_k - \phi_k(\omega)\big)\Big]^{\frac{1}{2}} \tag{3.10}$$

Obviously, the circular uncertainty case (3.9) is recovered by (3.10) after setting $A_k(\omega) = B_k(\omega)$.

Rectangular Uncertainties

Let $A_k(\omega) > 0$ and $B_k(\omega) > 0$ respectively represent the frequency-dependent half-width and the half-height of a rectangle centered at the origin of the complex plane as shown in Figure 3c. Consider the orientation-phase definition

$$\phi_k(\omega) := \arctan \frac{B_k(\omega)}{A_k(\omega)} \tag{3.11a}$$

and the associated phase sets

$$\Theta_{ak} := [0,\ \phi_k) \cup [\pi - \phi_k,\ \pi + \phi_k) \cup [2\pi - \phi_k, 2\pi] \tag{3.11b}$$

$$\Theta_{bk} := [\phi_k, \pi - \phi_k) \cup [\pi + \phi_k, 2\pi - \phi_k) \tag{3.11c}$$

The shaded and the plain areas in the Figure 3.3c corresponds to Θ_{bk} and Θ_{ak}, respectively. Then an analytical expression for the magnitude of any point on the boundary of the rectangular regions can be compactly written in the form

$$\gamma_k(\omega, \theta_k) = \begin{cases} \dfrac{A_k(\omega)}{|\cos(\theta_k)|} & \text{for } \theta_k \in \Theta_{ak} \\ \dfrac{B_k(\omega)}{|\sin(\theta_k)|} & \text{for } \theta_k \in \Theta_{bk} \end{cases} \tag{3.12}$$

3.3. The Critical Direction for a Characteristic Polynomial

The critical direction theory proposed for rational systems in [Latchman *et al.*, 1997] can be readily modified for the case of polynomials through the adoption of the definitions given below.

(i) The *Critical Direction*

$$d(j\omega) := -\frac{p_o(j\omega)}{|p_o(j\omega)|} := e^{j\theta_c(\omega)} \tag{3.13}$$

is the unit vector which defines the direction from the nominal point $p_o(j\omega)$ towards the origin. The critical direction is uniquely identified by the *critical phase angle* $\theta_c(\omega)$.

(ii) The *Uncertainty Template* (or *value set*)

$$\mathcal{T}(\omega) := \{ p(j\omega,\boldsymbol{q}) \in C \mid p(j\omega,\boldsymbol{q}) = p_o(j\omega) + \delta(j\omega,\boldsymbol{q}),\ \ q \in Q \ \} \qquad (3.14)$$

(iii) The *Critical Template*

$$\mathcal{T}_c(\omega) := \{ p(j\omega,\boldsymbol{q}) \in \mathcal{T}(\omega) \mid p(j\omega,\boldsymbol{q}) = p_o(j\omega) + r\,d(j\omega), \text{ for some } r \in \Re^+ \} \qquad (3.15)$$

(iv) The *Critical Perturbation Radius*

$$\rho_c(\omega) := \max_{r \in \Re^+} \{ r \mid z = p_o(j\omega) + r\,d(j\omega) \in \mathcal{T}_c(\omega) \} \qquad (3.16)$$

(v) The *Nyquist Robust Stability Margin*

$$k_N(\omega) := \frac{\rho_c(\omega)}{|p_o(j\omega)|} \qquad (3.17)$$

Note that the Nyquist robust stability margin $k_N(\omega)$ defined in (3.17) is a metric that characterizes the distance on the Nyquist plane to the point of instability (*i.e.*, the point $1+j0$). In a complementary fashion, the parametric robust stability margin $\alpha^*(\omega)$ defined in (3.8) characterizes the "distance" on the parameter space $\boldsymbol{Q}$ to the limit of stability.

3.4. Main Results

This section develops a technique for deriving analytical expressions for the parametric robust stability margin. The approach makes use of the critical-direction theory elements presented in Section 3.2. The developments are specialized here for the case of uncertain polynomial systems.

Lemma 3.1. *Consider the uncertain characteristic polynomial (3.4), and assume that nominal stability is attained under unity feedback, and that the critical templates*

$\mathcal{T}(\omega)$ are convex at all frequencies. Then the uncertain system remains stable for all parameter uncertainties $q \in Q$ if and only if

$$k_N(\omega) < 1 \quad \forall \omega \tag{3.18}$$

Proof. The proof follows from an application of the zero exclusion principle and is analogous to the development given in [Latchman *et al.*, 1997] for transfer functions, and details are omitted for brevity. ∇

Note that Lemma 3.1 is quite general and holds for multiaffine, and complex and/or real uncertainties. The major issue is the calculation of the critical radius $\rho_c(\omega)$ needed for the characterization of $k_N(\omega)$ in (3.17); however, as shown below in Theorem 3.1, the computation of the critical radius for complex affine parametric uncertainties is particularly straightforward.

Theorem 3.1. *Consider the uncertain system (3.3) with the uncertainty description (3.2a)-(3.2c), and its corresponding closed-loop uncertain characteristic polynomial (3.7). Then the critical perturbation radius is given by the expression*

$$\rho_c(\omega) = \sum_{k=1}^{m} \left| p_k(j\omega) \right| \ \gamma_k\left(\omega, \theta_c(\omega) - \theta_{p_k}(\omega)\right) \tag{3.19}$$

Proof. The points $p(j\omega, q)$ on the value set $\mathcal{T}(\omega)$ that belong to the critical template $\mathcal{T}_c(\omega)$ must satisfy the condition

$$p(j\omega, q) = p_o(j\omega) + \delta(j\omega, q) = p_o(j\omega) + rd(j\omega) \tag{3.20}$$

or

$$\sum_{k=1}^{m} p_k(j\omega)\, q_k(j\omega) = re^{j\theta_c(\omega)} \tag{3.21}$$

Due to the radial convexity of the uncertainty set, it suffices to consider only boundary uncertainties $\partial q_k(\omega)$; then, utilizing (3.2c) the preceding equality can be rewritten in the form

$$\sum_{k=1}^{m} \left| p_k(j\omega) \right| \gamma_k(\omega, \theta_k) e^{j\left(\theta_k + \theta_{p_k} - \theta_c(\omega)\right)} = r \tag{3.22}$$

From the definition (3.16) it follows that $\rho_c(\omega)$ is equal to the maximal value of r in (3.22) over all possible uncertainty phases $0 \le \theta_k \le 2\pi$, *i.e.*,

$$\rho_c(\omega) = \max_{0 \le \theta_k \le 2\pi} \sum_{k=1}^{m} \left| p_k(j\omega) \right| \gamma_k(\omega, \theta_k) \; e^{j\left(\theta_k + \theta_{p_k} - \theta_c(\omega)\right)} \tag{3.23}$$

Clearly, the maximum is obtained for $\theta_k = \theta_c(\omega) - \theta_{p_k}(\omega)$, leading to the expression (3.19). ∇

Theorem 3.2. *Consider the uncertain system (3.3) with the uncertainty description (3.2a)-(3.2c), and the corresponding critical perturbation radius* $\rho_c(\omega)$. *Then the parametric robust stability margin is given by*

$$\alpha^*(\omega) = \frac{\left| p_o(j\omega) \right|}{\rho_c(\omega)} \tag{3.24}$$

Proof. From equation (3.4)

$$p(j\omega, \boldsymbol{q}) = p_o(j\omega) + \sum_{k=1}^{m} p_k(j\omega) \; q_k(j\omega), \quad \boldsymbol{q} \in Q \tag{3.25}$$

and

$$p(j\omega, \alpha \boldsymbol{q}) = p_o(j\omega) + \alpha \sum_{k=1}^{m} p_k(j\omega) \; q_k(j\omega), \quad \boldsymbol{q} \in Q \tag{3.26}$$

Note that the map (3.25) yields the uncertainty template $\mathcal{T}(\omega)$ defined in (3.14). As for the question of stability, note that only those uncertainties that give rise to points $p(j\omega, \boldsymbol{q})$ that are aligned with the critical direction are relevant. Then, using the definition (3.13) for the critical direction, all the points of relevance for the stability analysis must be of the form

$$p(j\omega, \boldsymbol{q}) = p_o(j\omega) + r d(j\omega) \tag{3.27}$$

Equating (3.25) and (3.27) yields

$$\sum_{k=1}^{m} p_k(j\omega)\ q_k(j\omega) = r\,d(j\omega) \tag{3.28}$$

Note that if $q_k \in Q$ is such that $p(j\omega, \boldsymbol{q})$ is aligned with the critical direction, then for any scalar $\alpha > 0$, (3.26) implies that $p(j\omega, \alpha \boldsymbol{q})$ also is aligned with the critical direction. Hence, making use again of the definition of the critical direction given in (3.13), and from (3.25) and (3.27) it is concluded that

$$p(j\omega, \alpha \boldsymbol{q}) = p_o(j\omega) + \alpha\, r\, d(j\omega) = d(j\omega)\left(\alpha\, r - |p_o(j\omega)|\right) \tag{3.29}$$

from which if clearly follows that

$$p(j\omega, \alpha \boldsymbol{q}) = 0 \ \Leftrightarrow\ \alpha = \frac{|p_o(j\omega)|}{r} \tag{3.30}$$

Since by definition the maximum possible value of r is $\rho_c(\omega)$,it follows that $\alpha^*(\omega)$ is obtained when $r = \rho_c(\omega)$ and hence $\alpha^*(\omega) = |p_o(j\omega)| / \rho_c(\omega)$. ∇

Combining Theorem 3.1 and Theorem 3.2, it follows that the parametric robust stability margin (3.8) can be written in the form

$$\alpha^*(\omega) \ = \ \frac{|p_o(j\omega)|}{\sum_{k=1}^{m} |p_k(j\omega)|\ \gamma_k\left(\omega, \theta_c(\omega) - \theta_{p_k}(\omega)\right)} \tag{3.31}$$

Hence, (3.31) gives an analytical expression for the parametric robust stability margin for all complex element-wise parametric uncertainties that are closed under contraction of the form (3.2a)-(3.2c) for systems with the affine structure (3.3).

3.5. Parametric Robust Stability Margins for Highly Structured Uncertainties

The connection between the Nyquist robust stability margin $k_N(\omega)$ and the parametric robust stability margin $\alpha^*(\omega)$ is the reciprocal relationship

$$\alpha^*(\omega) \ = \ \frac{1}{k_N(\omega)} \tag{3.32}$$

which follows from Theorem 3.2 and the definition (3.17). Invoking Lemma 3.1, it then concluded that the closed-loop system is robustly stable if and only if $\alpha^*(\omega) < 1,\ \forall \omega$.

This result holds for all element-wise complex parametric uncertainty descriptions that are closed under contraction. This section focuses on obtaining analytical expressions for parametric robustness margin for the three specific types of uncertainty descriptions depicted in Figure 3.3

Theorem 3.3 *Consider the uncertain system (3.3) with the characteristic polynomial (3.7), and the circular, elliptical, and rectangular complex parametric uncertainty descriptions whose element-wise magnitudes are respectively given by (3.9), (3.10) and (3.12). The robust stability margins for each of these descriptions are given by the analytical expressions:*

(3.i) For the case the circular uncertainty description

$$\alpha^*(\omega) = \frac{|p_o(j\omega)|}{\sum_{k=1}^{m} A_k(\omega)\,|p_k(\mathrm{j}\omega)|} \tag{3.33a}$$

(ii) For the case the elliptical uncertainty description

$$\alpha^*(\omega) = \frac{|p_o(j\omega)|}{\sum_{k=1}^{m}\left[A_k(\omega)^2\cos^2\left(\theta_c(\omega)-\theta_{p_k}(\omega)-\phi_k(\omega)\right)+B_k(\omega)^2\sin^2\left(\theta_c(\omega)-\theta_{p_k}(\omega)-\phi_k(\omega)\right)\right]^{\frac{1}{2}}|p_k(\mathrm{j}\omega)|} \tag{3.33b}$$

(iii) For the case the rectangular uncertainty description

$$\alpha^*(\omega) = \frac{|p_o(j\omega)|}{\sum_{k=1}^{m}\left[\frac{A_k(\omega)|p_k(\mathrm{j}\omega)|}{\left|\cos\left(\theta_c(\omega)-\theta_{p_k}(\omega)\right)\right|}\delta_{ak}+\frac{B_k(\omega)|p_k(\mathrm{j}\omega)|}{\left|\sin\left(\theta_c(\omega)-\theta_{p_k}(\omega)\right)\right|}\delta_{bk}\right]} \tag{3.33c}$$

where $\delta_{ak}=1$ *if* $\theta_k\in\Theta_{ak}$, *otherwise* $\delta_{ak}=0$, *and where* $\delta_{bk}=1$ *if* $\theta_k\in\Theta_{bk}$, *otherwise* $\delta_{bk}=0$.

Proof. Equations (3.28b)-(3.28a) simply follow by substituting the expressions for the phase-dependent uncertainty magnitudes (3.9), (3.10), and (3.12) into the analytical expression (3.27) for the parametric robust stability margin. ∇

The generalized structured singular value approach [Chen *et al.,* 1995b] has previously been used to derive the result (3.33a) for the case of circular uncertainties. In principle, the general formalism presented in [Chen *et al.,* 1995b] can also be used to derive the results (3.33b) and (3.33c); however, the critical direction theory extracts exact analytical results through very simple and straightforward geometric arguments.

3.6 Connections with the Classical M-Δ Formalism

In this section expression (3.31) for the parametric robust stability margin is rederived utilizing the framework of the classical $\boldsymbol{M-\Delta}$ formalism where robust stability is characterized by a determinantal constraint. The approach follows well-established developments, except that we simplify the derivations by invoking a critical-direction concept, namely, that a specific orientation in the Nyquist plane is the only direction of relevance for stability analysis

Lemma 3.2. *The unity negative-feedback configuration of the SISO uncertain system (3.3) with affine uncertainty elements can be transformed into an equivalent MIMO* $\boldsymbol{M-\Delta}$ *structure via a linear fractional transformation* [Zhou *et al.,* 1996] *where the uncertainty matrix* $\Delta(s)$ *is diagonal and the structural matrix* $M=(M_{ik})$ *is of rank-one and has rational elements given by* $M_{ik}(s)=-\dfrac{p_k(s)}{p_o(s)}$, $i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, m.$

Proof. The proof is an exercise in block-diagram algebra. Details are given in the Appendix. ∇

For the MIMO $\boldsymbol{M-\Delta}$ structure in question it is typical to define the parametric robust stability margin in terms of a determinantal condition, as follows

$$\alpha^*(\omega) \;:=\; \min_{\alpha \,\in\, \Re^+} \left\{ \alpha \;\middle|\; \det\bigl(I + \alpha M(\mathrm{j}\omega)\Delta(\mathrm{j}\omega)\bigr) = 0 \right\} \tag{3.34}$$

which is formally analogous to the definition of the parametric robust stability margin for a SISO system given in (3.8). In fact, the theorem below shows that the definitions (3.8) and (3.34) lead to identical expressions for the parametric robust stability margin for the case where the uncertainties appears affinely in the model.

Theorem 4. *Consider the system (3.3) with its associated affine uncertainty description (3.2) and suppose that the nominal system is stable under unity negative-feedback, and that the nominal and the uncertain systems have the same number of open-loop unstable poles. Then the parametric robust stability margin at each frequency is given by*

$$\alpha^*(\omega) = \frac{|p_o(j\omega)|}{\sum_{k=1}^{m} \left|\gamma_k\left(\theta_c(\omega) - \theta_{p_k}(\omega) - \pi\right)\right| |p_k(j\omega)|} \tag{3.35}$$

Proof. From Lemma 3.1, the uncertain system with unity negative feedback can be transformed into an equivalent MIMO $M - \Delta$ structure. Using standard arguments, the robust stability condition for the transformed MIMO system is given by the determinantal condition

$$\det\left(I - M(j\omega)\Delta(j\omega)\right) \neq 0 \quad \forall \omega \tag{3.36}$$

Introducing the parametric robust stability margin $\alpha \in \Re^+$ interpreted as a blow-up factor for the uncertainties, the determinantal condition is rewritten as $\det\left(I - \alpha M(j\omega)\Delta(j\omega)\right) \neq 0$. Since from Lemma 3.1, matrix M is of rank one, it is straightforward to establish the inequality

$$\det\left(I - \alpha M(j\omega)\Delta(j\omega)\right) = 1 + \frac{\alpha}{p_o(j\omega)}\left[\sum_{k=1}^{m} p_k(j\omega) q_k(j\omega)\right] \tag{3.37}$$

Since the aim is to find the minimum scalar $\alpha \in \Re^+$ that makes (3.37) identically zero, equating (3.37) to zero and solving for α yields

$$\alpha = \frac{\left|p_o(j\omega)\right| e^{j(\theta_{p_o}+\pi)}}{\sum_{k=1}^{m}\left|p_k(j\omega)\right| \gamma_k(\omega,\theta_k) e^{j(\theta_k+\theta_{pk})}} \tag{3.38}$$

Clearly, the minimum real non-negative value of (3.37) will be realized when all terms in the summation in the denominator (3.37) are aligned with respect to each other and with respect to the numerator, *i.e.*, using the notation $p_o(j\omega) = \left|p_o(j\omega)\right| e^{j\theta_{p_o}}$ and recalling the fact that the uncertainty phase is θ_k unrestricted, the minimum is attained when

$$\theta_k = \theta_{p_o}(\omega) - \theta_{p_k}(\omega) + \pi, \ k = 1, 2, \cdots, m \tag{3.39}$$

From the definition (3.13) it follows that $\theta_{p_o}(\omega) = \theta_c(\omega) - \pi$, hence the minimizer is realized at

$$\theta_k = \theta_c(\omega) - \theta_{p_k}(\omega), \ k = 1, 2, \cdots, m \tag{3.40}$$

Substituting (3.40) into (3.38) gives the minimal value $\alpha = \alpha^*(\omega)$ which is of the form (3.30). ∇

3.7 Conclusions

In this chapter we demonstrated the significance and ease of application of the critical direction theory to the complex affine uncertainty problem. The development is simple and intuitive , and for the special cases of interest recovers the exact results by the previous approaches. The case of multiaffine, complex and/real uncertainties may also be handled using the critical direction arguments. However, for all these cases the computation of the critical radius remains the major theoretical challenge; fortunately, for a number of uncertainty descriptions, such as the ones considered in this chapter , it is possible to obtain analytical expressions for the critical radius.

Nomenclature for Chapter 3

s	Laplace variable
$\boldsymbol{q}$	Uncertainty vector
$n(j\omega,\boldsymbol{q})$	Uncertain numerator polynomial
$d(j\omega,\boldsymbol{q})$	Uncertain denominator polynomial
$p(j\omega,\boldsymbol{q})$	Uncertain characteristic polynomial
$d_c(j\omega)$	Critical direction in the complex plane
$k_N(\omega)$	Nyquist robust stability margin
$k_m(\omega)$	Multivariable robust stability margin

Greek Letters

$\alpha^*(\omega)$	Blow up factor
ω	Frequency
$\rho_c(\omega)$	Critical perturbation radius
$\mu(\omega)$	Structured singular value
$\bar{\sigma}(.)$	Maximum singular value
$\rho(.)$	Spectral radius
$\mathcal{T}(\omega)$	Eigen template
$\mathcal{T}_c(\omega)$	Critical eigen template

CHAPTER 4

VARIABLE STRUCTURE CONTROL DESIGN FOR REDUCED CHATTER IN UNCERTAIN STATE DELAY SYSTEMS

4.1 Introduction

The study of uncertain state-delay and input-delay systems has received much attention by researchers in the past decade [*cf.* Zaveri and Jamshidi, 1987 and Hale and Lunel, 1993 for background and extensive references]. One reason for this interest is that time delay is encountered in various engineering systems and can be the cause of instability. Another reason is that practical control systems unavoidably include uncertainty or disturbances due to modeling or measurement errors and other approximations. Several authors have investigated the problem of stabilization of uncertain state-delay systems. The main strategies for stabilization of delay systems include designing state or output feedback controllers (using pole assignment, Lyapunov or LQ theory), and variable structure controllers (VSC). Cheres *et al.*, [1989] construct a min-max controller from the knowledge of the upper bound on the delay. Shen *et al.*, [1991] propose a memoryless linear state feedback based on Riccati equation approach [also see Wang and Lin, 1988]. Phoojaruenchanachai and Furuta [1992] also construct a memoryless state feedback, but consider a larger class of systems. Trinh and Aldeen, [1994] consider interconnected systems and propose memoryless state feedback controllers to stabilize the uncertain state-delay systems. Niculescu *et al.*, [1996] design memoryless state feedback controllers to stabilize an uncertain state delay system with constrained input via ricatti approach.

Recently, sliding mode control has been employed as a tool for stabilization of uncertain state-delay systems. The sliding mode control possesses well-known features that make it very attractive for control systems. These include fast response, insensitivity

to parameter variations, and decoupling design procedure, among others. Recently Shyu and Yan [1993] and Oucheriah [1995] have used VSC to guarantee the stability of uncertain delay systems, deriving sufficient conditions that depend on the size of the delay. However, the Shyu-Yan method suffers from the disadvantage that the control synthesis procedure involves unknown matrices; hence the control law can not be implemented. Oucheriah proposes a dynamic switching-surface control scheme based on pole-assignment. Both of the above approaches involve strong discontinuous control across the switching surface to overcome the effects of the uncertainty, and hence suffer from severe chattering. Chattering is undesirable because it might excite the unmodeled high-frequency components of the system, and it may lead to premature wear or tear of the actuators. The recent technique of perturbation compensation has been shown to provide effective chatter suppression for delay-free systems [Kim, 1992; Chan, 1996]. This method exploits the specific structure of VSC to estimate and compensate for the effect of perturbation, and thus results in reduction or removal of chattering.

This chapter proposes a VSC design methodology to robustly stabilize an uncertain state-delay system with nonlinear and possibly mismatched uncertainties, utilizing the concept of perturbation compensation. Sufficient robust stability conditions are derived which are independent of the size of the delay; furthermore no matching conditions on the uncertainty are assumed. The main advantages of the proposed approach are that *(i)* a realizable control law is obtained which does not need the bounds on the system uncertainty, *(ii)* chattering reduction/elimination is achieved, *(iii)* delay-independent sufficient robust stability conditions which can accommodate uncertainty in the delay are derived, and *(iv)* guidelines are given for the design of the control matrix involved in the specification of the switching function. Finally the results are illustrated with an open-loop unstable example.

4.2 Sliding Mode Control Design

Consider an uncertain state-delay system subject to uncertainty and/or external perturbations governed by the equations

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{A}_d\boldsymbol{x}(t-h) + \boldsymbol{B}\boldsymbol{u}(t) + f(\boldsymbol{x}(t),t) + f_d(\boldsymbol{x}(t-h),t) \\ \boldsymbol{x}(\theta) &= \boldsymbol{\phi}(\theta) \quad , \quad \theta \in [-h,\ 0] \end{aligned} \tag{4.1}$$

where $\boldsymbol{x}(t) \in R^n$ is the state vector which is assumed measurable and has the initial state $\boldsymbol{x}(0) = \boldsymbol{x}_0$, $\boldsymbol{u}(t) \in R^m$ is the input vector; $\boldsymbol{A} \in R^{n\times n}$ and $\boldsymbol{B} \in R^{n\times m}$ are constant matrices; $\boldsymbol{\phi}(t)$ is a continuous vector-valued initial function; and $h > 0$ is the time delay. The vector functions $f(\boldsymbol{x}(t),t) \in R^n$ and $f_d(\boldsymbol{x}(t-h),t) \in R^n$ represent nonlinear perturbations that depend on the current state $\boldsymbol{x}(t)$ and on the delayed state $\boldsymbol{x}(t-h)$ of the system, respectively. It is assumed that the modeling uncertainties satisfy the bounds

$$\begin{aligned} \|f(\boldsymbol{x}(t),t)\| &\le k\ \|\boldsymbol{x}(t)\| \\ \|f_d(\boldsymbol{x}(t-h),t)\| &\le k_d\|\boldsymbol{x}(t-h)\| \end{aligned} \tag{4.2}$$

where k and k_d are known positive real constants, and $\|\cdot\|$ is any vector norm or corresponding induced matrix norm. Notice that no matching assumptions are used. It should be noted that these bounds are required not for controller design, but for robust stability analysis.

Two definitions are useful for the ensuing developments. The matrix measure [Vidyasagar, 1993] is defined as the function

$$\mu : C^{n\times n} \to R : \mu(\boldsymbol{A}) = \lim_{\varepsilon\to 0^+} \frac{\|\boldsymbol{I} + \varepsilon\boldsymbol{A}\| - 1}{\varepsilon}$$

The matrix measure is also known as the *logarithmic norm*. For an excellent discussion on logarithmic norms, see [Strom, 1975].

Also, a class of matrix pairs $(\boldsymbol{A}, \boldsymbol{B})$ is introduced for the stabilization problem, defined through the set membership

$$\mathcal{U} := \left\{(\boldsymbol{A},\boldsymbol{B}) \middle| \exists\, F \in R^{n\times m} : \mu(\boldsymbol{A} + \boldsymbol{B}F) < 0\right\} \tag{4.3}$$

where $\mathcal{U}$ clearly represents the class of systems for which there exists a state-feedback law $u(t) = Fx(t)$ such that $\mu(A + BF) < 0$.

The objective is to design a control law that assures reachability of the sliding mode within a finite time and asymptotic stability of the system in the sliding mode, even in the presence of uncertainties and/or external disturbances. The design involves two major phases: The first phase is the selection of a switching surface manifold that has desirable dynamics, and the second phase is the determination of an appropriate control law that guarantees the existence and reachability of the sliding mode and the effective rejection of disturbances.

Switching Function and Control Design

This section discusses the selection of an appropriate switching function. In the sliding mode the system must not only be stable but also be robust to any kind of bounded perturbations as specified in (4.3). The procedure is as follows. First, select matrices F and F_d such that matrix $\overline{A} = A - BF$ is a stable matrix, and that matrix $\overline{A}_d = A_d - BF_d$ satisfies $\|\overline{A}_d\| \leq \|A_d\|$. At this point we anticipate that the role of F_d is to reduce the effect of delay, since the stability conditions derived later in this chapter involve $\|\overline{A}_d\|$. Using the matrices described above the system equation (4.1) can be rewritten in the form

$$\dot{x}(t) = \left(\overline{A} + BF\right)x(t) + \left(\overline{A}_d + BF_d\right)x(t-h) + Bu(t) + f(x(t),t) + f_d(x(t-h),t) \quad (4.4)$$

An integral switching function [Fernandez and Hedrik, 1987] is adopted in the modified form proposed by Shyu and Yan [1993] namely

$$s(t) = Cx(t) - \int_0^t \left\{C\overline{A}x(\tau) + C\overline{A}_d x(\tau - h)\right\} d\tau \quad (4.5)$$

where matrix $C \in R^{mxn}$ is a control design matrix that is chosen such that the product CB is nonsingular. The reason for choosing such a switching function instead of a traditional combination of the states is that, this choice results in a desirable switching dynamics. This will be clear after the design of the controller.

The goal is to find a VSC controller that drives the system to the switching surface and ensures a sliding mode condition. The controller structure adopted is of the form

$$u(t) = u_{eq}(t) + u_r(t) + u_p(t)$$

which contains three terms. The first term in turn is of the form

$$u_{eq}(t) = -\left(Fx(t) + F_d x(t-h)\right) \tag{4.6}$$

and thus represents the model-based equivalent control for the nominal system in the absence of perturbations, obtained by setting $\dot{s}(t) = 0$ [DeCarlo *et al.*, 1988]. The second term

$$u_r(t) = -(CB)^{-1}\left(Ks(t) + d\ \mathrm{sgn}[s(t)]\right) \tag{4.7}$$

is the reaching control law where the symmetric positive-definite matrix K and the scalar $d \geq 0$ are design parameters. The final term is intended to cancel the effect of perturbations on the dynamics of the switching function [Chan, 1996] and is given by

$$u_p(t) = -(CB)^{-1} T_p(t) \tag{4.8}$$

with

$$T_p(t) = \dot{s}(t) - CB\left(u_r(t) + u_p(t-T)\right) \tag{4.9}$$

where $T_p(t)$ is designed to reject the effects of the perturbation signal. In order to avoid an algebraic loop in calculation, a deliberately delayed value $u_p(t-T)$ is used. The value of the computational time delay T is chosen to be small. As long as the variations in the system dynamics are slower than the computational delay, this will be good approximation. However, it should be admitted that regardless of the value of T, $u_p(t) \neq u_p(t-T)$. However, as long as the estimation error is less than the actual perturbation, we only gain by using perturbation compensation. Also a finite difference scheme is used to estimate $\dot{s}(t)$. This could also be a source of errors if the measurements are noisy.

The ideal VSC approach assumes that it is possible to switch infinitely fast across the switching surface. The phenomenon of non-ideal but fast switching is labeled as

chattering [DeCarlo *et al.,* 1988]. Interestingly, the name comes from noise generated by the relays. Chattering is undesirable for two reasons: *(i)* It might excite the unmodeled high frequency plant dynamics which could lead to unforeseen dynamics. Note that in certain applications it is not of concern as the frequency of switching is far above the structural frequencies of the mechanical system [DeCarlo *et al.,* 1988]. However, in most applications this will be a major concern, *(ii)* It might lead to the premature wear and tear of the actuators. It should be noted here that it is the discontinuous control, which in turn leads to chattering, enables one to guarantee the robustness features of sliding mode control. In a sense, we gain robustness but pay for it with chattering.

To alleviate chattering, many approaches have been proposed in the literature. Here, we give a brief review of the past efforts. Although, they have not been used for sliding mode control of state-delayed systems, they are nevertheless useful as a concept. One of them is the "boundary layer approach" [see Slottine and Sastry, 1983] which uses the continuous control law and time varying sliding surfaces. Although this succeeds in eliminating chattering, it is no longer possible to assure the asymptotic stability of the system. Rather, one has to be content with uniform ultimate boundedness of the trajectories within a neighborhood of the origin depending on the boundary layer. Ambrosino *et al.*, [1984] cascaded a low-pass filter after the VSC to attenuate the high frequency switching component. In this case chattering is alleviated at the expense of robustness. Espana *et al.*, [1984] propose a zone in the vicinity of the switching surface in which the feedback gain is adjusted in order to avoid chattering. Morgan and Ozguner [1985] propose to decrease the control gain in order to minimize chattering, but the disadvantage is the increased time that it takes to achieve sliding mode. Chang *et al.*, [1990] proposed an adaptive alleviation algorithm which requires recursive prediction at each step. Their procedure involves computing the derivative of the states, but they assume that they are either available or can be computed without errors. Elmali and Olgac [1992] do an on-line estimation of the effect of the perturbations based on the measurements of the

state dynamics. They assume that the error in estimation is bounded within a scalar multiple of the estimation. Also see an experimental verification of their approach in Elmali and Olgac [1996]. Chan [1996] proposed an perturbation compensation approach utilizing the specific structure of the variable structure control. The approach seeks to estimate the effect of perturbations on the switching dynamics with the assumption that the relative error in the estimation of the switching dynamics is bounded and known. One of the major advantages of the perturbation compensation schemes is that one need not have *a priori* bounds on the uncertainties for controller design.

Note that with the application of the control law (4.6) - (4.9) and the choice of the switching function (4.5), the derivative function $\dot{s}(t)$ is given by

$$\dot{s}(t) = \boldsymbol{CB}\left(u_r(t) + u_p(t)\right) + \boldsymbol{C}\left(f(x(t),t) + f_d(x(t-h),t)\right)$$

$$\dot{s}(t) = -\left(Ks(t) + d\ \mathbf{sgn}[s(t)]\right) + \left[\boldsymbol{C}\left(f(x(t),t) + f_d(x(t-h),t)\right) - T_p\right]$$

It is clear that $\dot{s}(t)$ is driven by the estimation error, and the purpose of control is to counter the effect of perturbations on the switching surface. The usual approach is to overpower the uncertainty by discontinuous control. For example, in this case the value of d should be chosen such that $d > |\boldsymbol{C}|\left(k\ \|x(t)\| + k_d\|x(t-h)\|\right)$. In fact, a choice of this type is made by [Oucheriah, 1995]. For proving the reaching condition, this is indeed a clever selection of parameters. However, there are some significant problems with this choice. Firstly, we are assuming it is always the worst case that is affecting the system. Secondly, there is no method to estimate even conservatively, the norm of the states when the system oscillates near the switching surface. Hence, while we are guaranteed to suffer from chattering, we have no available worst case upper bound.

In contrast, the effect of $u_p(t)$ is to ease the burden on the discontinuous control by canceling the effect of perturbations on the switching dynamics. The discontinuous control is now utilized, to combat the residual error in estimation rather than the whole perturbation. Irrespective of the actual value of the perturbation, as long as the estimation

scheme works well, this scheme will outperform the conventional VSC. One point needs to be clarified here. Estimating $\dot{s}(t)$ is in fact, equivalent to saying that we are estimating the perturbation. Our contention here is that $\dot{s}(t)$ can be estimated from the measurable states of the system via (4.5). It goes without saying that we are introducing a new parameter here, namely the bound on the estimation error. Apart from the measurement error in the states, there is also the danger of noise. Since $s(t)$ depends on $x(t)$, it is assumed that proper filtering strategies will be employed in practice. In the worst case, one has to cut off the perturbation compensation and has to live with the chattering. By worst case, we mean that the computed perturbation signal is greater than the maximum norm of the allowed perturbation, *i.e.*,

$$\left|T_p(t)\right| < \left|C\left(f(x(t),t)+f_d(x(t-h),t)\right)\right| < \left|C\right|\left(k\left|x(t)\right|+k_d\left|x(t-h)\right|\right)$$

However, this decision can only be made depending on the application at hand. For theoretical development, in order to prove the reaching condition within a finite time, we need to assume certain error bounds on the derivative estimation. We assume that the relative error in the estimation of $\dot{s}(t)$ is bounded,

$$\left|\frac{\dot{s}(t)-\dot{s}_{est}(t)}{\dot{s}_{est}(t)}\right| < \delta$$

with the positive scalar bound δ being known. An estimate of δ can be obtained via simulation studies with appropriately assumed perturbations. Then with the choice $d(t) > \delta\dot{s}_{est}(t) + \varepsilon$, $\varepsilon > 0$, and K the reaching condition can be satisfied and in addition, the transient behavior can be shaped.

Theorem 4.1. *The uncertain delay system (4.1) with model-uncertainty and bounded measurement uncertainty given by (4.2) and (4.10) achieves the desired sliding mode within a finite time if the control law* $\boldsymbol{u}(t) = \boldsymbol{u}_{eq}(t) + \boldsymbol{u}_r(t) + \boldsymbol{u}_p(t)$ *is applied, where the individual components are given by (4.6)-(4.9).*

Furthermore, the sliding mode is attained within $t = t_r$, where

$$t_r \le \left(\frac{2}{\lambda_{\min}(K)}\right)\ln\left[1+\frac{\lambda_{\min}(K)}{\sqrt{2}\varepsilon}\left(\sqrt{s^T(0)s(0)}\right)\right]$$

and the upper bound on the norm of the switching surface is given by

$$|s(t)| \le \left[|s(0)|+\frac{\sqrt{2}\varepsilon}{\lambda_{\min}(K)}\right]e^{-\frac{\lambda_{\min}(K)}{2}t}-\frac{\sqrt{2}\varepsilon}{\lambda_{\min}(K)}$$

Proof: Choose a positive definite function $V(t) = \frac{1}{2}\, s(t)^T s(t)$. Then differentiating $V(t)$ along the trajectories of the system and using (4.5)-(4.10) yields:

$$\dot{V}(t) = s(t)^T\dot{s}(t) = s(t)^T\left[-Ks(t) - d\frac{s(t)}{\|s(t)\|} + \left(C\left(f(x(t),t)+f_d(x(t-h),t)\right)-T_p(t)\right)\right] \quad (4.11)$$

In the rest of the proof the time argument is dropped for convenience of notation. Noting that $\Delta\dot{s} = C\left(f(x(t),t)+f_d(x(t-h),t)\right)-T_p(t)$ and using the bound on the estimation error (4.10) the equation (4.11) can be used to derive a bound on $\dot{V}(t)$ as follows:

$$\dot{V}(t) = -s^TKs - ds^T sign(s) + s^T\Delta\dot{s}$$

We note that since $s(t)$ is a vector, $s^T sign(s) = |s|_1 > |s|_2$. Hence,

$$\dot{V}(t) \le -s^TKs - d|s| + |s||\Delta\dot{s}|$$

Since $d = \delta|\dot{s}_{est}(t)| + \varepsilon$, and $|\Delta\dot{s}| \le \delta|\dot{s}_{est}(t)|$

$$\dot{V}(t) \le \left(-\lambda_{\min}(K)\right)|s|^2 - \varepsilon|s|$$

$$\dot{V}(t) \le \left(-\lambda_{\min}(K)\right)V - \varepsilon\sqrt{V}$$

The above is a scalar differential equation and the solution is given by

$$\left|\sqrt{V(t)}\right| \le \left[\sqrt{V(0)}+\frac{\varepsilon}{\lambda_{\min}(K)}\right]e^{-\frac{\lambda_{\min}(K)}{2}t}-\frac{\varepsilon}{\lambda_{\min}(K)}$$

$$|s(t)| \le \left[|s(0)| + \frac{\sqrt{2}\varepsilon}{\lambda_{\min}(K)} \right] e^{-\frac{\lambda_{\min}(K)}{2} t} - \frac{\sqrt{2}\varepsilon}{\lambda_{\min}(K)}$$

Note that first term is always positive and decreasing, and the second term is a positive constant and hence, their difference will attain zero within a finite time. It is straightforward to compute the upper bound on the reaching time. A detailed proof is given in Appendix F. Thus the system is guaranteed to reach the sliding mode within a finite time since the reaching condition $s(t)^T \dot{s}(t) < 0$ for $s(t) \neq 0$ is satisfied. ∇

Remarks:

1. The rate at which the switching surface is reached is determined by the controller parameters K and d. Therefore by adjusting these parameters the designer may select an appropriate reaching-transient behavior.
2. Chattering in the input variable is unavoidable for any positive value of d, because $u_r(t) \cong \pm d$ when the system oscillates around $s = 0$. To reduce chattering, the parameter d must be chosen as small as possible. In some cases $\dot{s}(t)$ is measured directly, therefore d can be set equal to zero without any loss. In other cases $\dot{s}_{\text{est}}(t)$ can be compared with simulated $\dot{s}(t)$ (for a given set of perturbations) to aid in choosing a suitable the parameter d. Alternatively, one can choose to set $d \approx 0$ and specify a large K to overpower the estimation errors if any are present. However in the last case the penalty to be paid is high initial controller gains.

4.3 Asymptotic Stability with Perturbation Compensation

Once in the sliding mode the system equations reduce to

$$\dot{x}(t) = \overline{A}x(t) + \overline{A}_d x(t-h) + f(x(t),t) + f_d(x(t-h),t) - B(CB)^{-1} T_p(t) \quad (4.12)$$

which is obtained from (4.4) after substituting (4.6)-(4.8). The intent now is to derive conditions that establish the asymptotic stability of system (4.12). First, we note that if perturbation compensation is not employed, then (4.12) reduces to

$$\dot{x}(t) = \overline{A}x(t) + \overline{A}_d x(t-h) + f(x(t),t) + f_d(x(t-h),t) \tag{4.13}$$

With perfect perturbation compensation, the system equations become

$$\dot{x}(t) = \overline{A}x(t) + \overline{A}_d x(t-h) + \gamma\left(f(x(t),t) + f_d(x(t-h),t)\right) \tag{4.14}$$

where $\gamma = \left[I - B(CB)^{-1}C\right]$

There are basically two approaches in the literature for deriving conditions for the asymptotic stability of these systems. One is to follow Lyapunov-type of arguments [Hale and Lunel, 1993], and the other is to use norm-based inequalities [Mori *et al.*, 1981]. Both approaches result in sufficient-only stability conditions.

Lemma 4.1. *The delay system (4.13) with nonlinear uncertainties (4.2) is asymptotically stable if the following condition is satisfied*

$$\mu(\overline{A}) + \|\overline{A}_d\| + k + k_d < 0 \tag{4.15}$$

Proof. See Appendix B. ∇

Theorem 4.2 *The delay system (4.14) with nonlinear uncertainties (4.2) is asymptotically stable if the following condition is satisfied*

$$\mu(\overline{A}) + \|\overline{A}_d\| + k\|\gamma\| + k_d\|\gamma\| < 0 \tag{4.16}$$

Proof. The proof is identical to the proof of the previous lemma. ∇

Remarks

1. It is possible that different choices of a given norm and the corresponding matrix measure give different conclusions about stability. This is consistent with the fact that the conditions derived are only sufficient.

2. In order to improve the sharpness of the stability condition, a similarity transformation $z(t) = T^{-1}x(t)$ can be introduced so that the system equation (4.14) becomes

$$\dot{z}(t) = T^{-1}\bar{A}Tz(t) + T^{-1}\bar{A}_d T z(t-h) + \gamma'(f(x(t),t) + f_d(x(t-h),t)) \quad (4.17)$$

where $\gamma' = T^{-1}\left(I - B(CB)^{-1}C\right)$. In this case the stability condition reads

$$\mu\left(T^{-1}\bar{A}T\right) + \left\|T^{-1}\bar{A}_d T\right\| + k_d\|T\|\left\|\gamma'\right\| + k\|T\|\left\|\gamma'\right\| < 0 \quad (4.18)$$

The usual approach is to find a transformation matrix T to diagonalize $T^{-1}\bar{A}T$.

4. Under perfect estimation and the assumption of matched uncertainties [Shyu and Yan, 1993] the perturbation error is $(f + f_d) + Bu_p = 0$. The stability condition then reduces to $\mu\left(T^{-1}\bar{A}T\right) + \left\|T^{-1}\bar{A}_d T\right\| < 0$. Of course, there is a penalty for imperfect estimation and this is reflected in the reduction of the size of the perturbations the system can withstand and still remain stable.

5. The conditions (4.15) and (4.16) are delay independent. This could be advantageous in cases where delay-dependent stability conditions cannot assure stability. The example in the chapter illustrates this fact. However, since all conditions reported to date in the literature (including this one) are sufficient only, no general claims can be made. It is also possible to derive delay-dependent conditions for the problem considered in the chapter.

Optimal Choice of C

The stability condition (4.18) and the condition derived in Shyu and Yan [1993] involves the matrix $\gamma' = T^{-1}\left(I - B(CB)^{-1}C\right)$. Then the matrix C is chosen to minimize $\left\| T^{-1}\left(I - B(CB)^{-1}C\right) \right\|_F$. There appears to be no documented systematic method for optimizing the left hand side of (4.18) (or similar stability conditions for other approaches) over all allowable matrices T and C. To get the tightest possible stability condition for the result in Shyu and Yan [1993] the matrix C must be chosen to minimize the norm of γ'. We propose a new systematic approach to design matrix C.

The following definition and inequalities concerning Frobenius norm are useful. Let $A \in \Re^{m\times n}$, and let $\| \ \|_F$, and σ_i^2 represent the Frobenius norm and 'i'th singular value, respectively. Then [Golub and VanLoan, 1989]

$$\|A\|_F := \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2}.$$

The following relationships are useful

$$\begin{aligned} \|A\|^2_F &= \sigma_1^2 + \cdots + \sigma_p^2; \quad p = \min\{m,n\} \\ \|A\|_2 &\le \|A\|_F \le \sqrt{n}\|A\|_2 \end{aligned} \tag{4.19}$$

Lemma 4.2. Let $N \in \Re^{n\times m}$ and $M \in \Re^{n\times n}$. Then the best approximate solution to the problem

$$\min_{X \in \Re^{m\times n}} \| M - NX \|_F$$

is $X_{\text{opt}} = \left(N^T N\right)^{-1} N^T M$.

Proof: See Barnett [1971].

The definition of X_{opt} implies that for all X, $X \in \Re^{m\times n}$

either

$$\| M - NX \|_F > \| M - NX_{\text{opt}} \|_F$$

or

$$\| M - NX \|_F = \| M - NX_{\text{opt}} \|_F \text{ and } \|X\| \geq \|X_{\text{opt}}\|$$

It is instructive to note that $\left(N^T N\right)^{-1} N^{\mathrm{T}}$ is the Moore-Penrose generalized inverse of matrix N.

Theorem 4.3. *The best approximate choice of matrix C for the following problem*

$$\min_{C \in \Re^{mxn}} \| T^{-1}\left(I - B(CB)^{-1}C\right) \|_F$$

is $C_{\text{opt}} = \alpha\left(B^T T^{-T} T^{-1}\right)$, where α is a scalar.

Proof. Define the auxiliary matrix $X = (CB)^{-1}C$. The minimization problem reduces to

$$\min_{X \in \Re^{mxn}} \| T^{-1}(I - BX) \|_F$$

Substituting $M = T^{-1}$ and $N = T^{-1}B$ in Lemma 4.2 the best approximate solution is found to be

$$X_{opt} = \left(B^T T^{-T} T^{-1} B\right)^{-1} B^T T^{-T} T^{-1} = \left(C_{\text{opt}} B\right)^{-I} C_{\text{opt}}$$

It is obvious that $C_{\text{opt}} = B^T T^{-T} T^{-1}$. Note that multiplying by a scalar α does not change the value of the functional being minimized, hence $C_{\text{opt}} = \alpha\left(B^T T^{-T} T^{-1}\right)$ ∇

Remarks:

1. Note that Theorem 4.3 gives the optimal solution in the sense of the Frobenius norm. Using (4.19), it is easy to see that for the case where $B \in \Re^{n\times 1}$, the choice $C_{\text{opt}} = \alpha\left(B^T T^{-T} T^{-1}\right)$ is optimal in the sense of the 2-norm as well.

2. The Matrix C enters in the last three terms of the stability condition (4.18). An alternative method for finding the optimal matrix C is to carry out a numerical optimization. However, noting that the contribution from the last term will be small for a well performing estimation strategy, the last term can be ignored for analysis purposes. Then, from triangular inequality it is easy to verify that the

result of Theorem 4.3 provides a sub optimal upper bound in the sense of the Frobenius norm.

Implementation Issues

1. The estimation error depends on the approximation scheme used. Although there is no *a priori* information on the estimation error, different estimation strategies can be compared by assuming appropriate perturbations and tracking the estimation error.
2. Also, note that the main challenge in the proposed approach is that of approximating $\dot{s}(t)$ by $\dot{s}_{est}(t)$. In general, a finite difference estimate is used. However, this may cause a problem near the switching surface where some chattering combined with a small step length may contribute to large estimation errors. Also, there is the unavoidable nuisance of noise, and the danger of differentiating it. Hence it is necessary to employ filtering. Although the switching dynamics with perturbation compensation is complex, one can still perform a low pass filtering on $s(t)$ for the purpose of estimating $\dot{s}_{est}(t)$ by finite difference [Elmali and Olgac, 1996]. Note that the actual value of $s(t)$ can be used elsewhere in the controller. Hence, this does not destroy the guarantee of asymptotic stability. The cutoff frequency for the filter can be chosen based on the maximum frequency component that we wish to process in the state dynamics. For example, if temperature in a reactor is the state variable that is used in the construction of the switching function, a cut-off frequency of 100 Hz is acceptable, whereas if pressure in a flowing line is the variable then one needs to go for higher frequencies. With the advances in digital signal processing these decisions can be made scientifically depending on the application at hand.

4.4 Example

Consider the uncertain time-delay system studied by Shyu and Yan [1993]

$$\dot{x} = \begin{bmatrix} 1 & 1.5 \\ 0.3 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} x(t-h) + \begin{bmatrix} 10 \\ 1 \end{bmatrix} u + f(x(t),t) + f_d(x(t-h),t), \quad t > 0$$

The delay is $h = 2$, and the unknown non-linear perturbations are of the form

$$f(x(t),t) = 0.2x(t)\sin(x(t)) \text{ and } f_d(x(t-h),t) = 0.1x(t-h)\sin(x(t-h))$$

The design objective is to asymptotically stabilize the uncertain system. The initial condition is specified as $x(t) = [1,3]^T \; -h \le t \le 0$. Note that for the delay considered, the sufficient stability condition of Shyu and Yan [1993] is violated.

Also note that the homogeneous part of the system is unstable because A has a positive eigenvalue $\lambda = 1.1432$. Choosing the same control matrices investigated in Shyu and Yan (1993), namely $F = [0.3 \quad 0.15]$, $F_d = [0 \quad -0.1]$ leads to

$$\bar{A} = \begin{bmatrix} -2 & 0 \\ 0 & -2.15 \end{bmatrix}, \quad \bar{A}_d = \begin{bmatrix} 0 & 0 \\ 0 & -0.1 \end{bmatrix}$$

The particular choice F adopted makes $\bar{A}$ diagonal (and hence $T = I$), and the choice of F_d ensures $\|\bar{A}_d\| < \|A_d\|$. As per Theorem 4.3, $C_{\text{opt}} = \alpha \left(B^T T^{-2}\right)$. In this design we choose $K = 2$, and $d = 0$. Then choosing $\alpha = 0.1$ leads to $C = [1 \quad 0.1]$. If a positive value for d is chosen, it will result in some chattering.

From (4.16) we can find the maximum tolerable estimation error. For the 1-norm and infinity norm we find that asymptotic stability is assured if $\varepsilon \le 1.4445$, while for the 2-norm asymptotic stability is assured if $\varepsilon \le 1.6085$. The delay introduced in the computation of $u_p(t)$ is 0.01 seconds.

The simulation results are presented in Figures 4.1 - 4.4. Figure 4.1 shows the evolution of state norm as a function of time, reflecting the asymptotic stability of the system. The switching function and its evolution towards the sliding mode is shown in Figure 4.2. The sliding mode is reached rather quickly, and most importantly, a chattering-

free behavior is observed. Figure 4.3 shows that the input variable is also smooth and free of chattering. The jump in the input around t = 2 reflects the fact that control variable is responding to the delayed states coming into effect. Figure 4.4 shows the evolution of the estimation error as a function of time, which also reaches the value of zero asymptotically. From Figure 4.4, it is obvious that the estimation error in the sliding mode is much less than 1.6085, thus implying robust stability via equation (4.16). In this example, there is no chattering, and the system is asymptotically stable.

4.5 Conclusions

This chapter considered the robust stabilization of uncertain time-delay systems with mismatched nonlinear uncertainties. A practically implementable control design methodology is proposed and an illustrative design example is given. The present method can significantly reduce or completely eliminate chattering as compared to previous approaches.

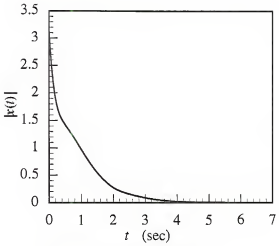

Figure 4.1. State norm Vs Time

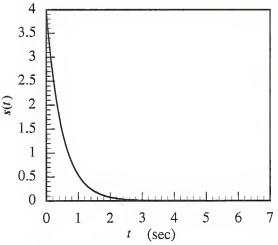

Figure 4.2. Switching function Vs Time

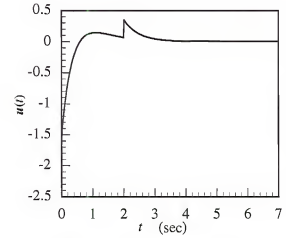

Figure 4.3. Control input Vs Time

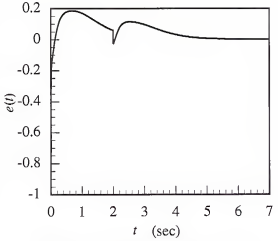

Figure 4.4. Compensation error Vs Time

Nomenclature for Chapter 4

A	State matrix
A_{d}	State delay matrix
B	Control matrix
C	Switching function matrix
d	Magnitude of discontinuous control
f	State dependent nonlinear perturbation
f_d	Delayed state dependent nonlinear perturbation
F	State feedback matrix
F_d	Delayed state feedback matrix
h	delay
K	Positive definite matrix
k	Bound on state dependent uncertainty
k_d	Bound on state delay dependent uncertainty
$s(t)$	Switching function
T	Transformation matrix
T_p	Estimated perturbation signal
$u(t)$	Control input
$x(t)$	State
$z(t)$	Integral equation with deviating argument

Greek Letters

δ	Bound on the derivative estimation error
ε	Derivative estimation error in the sliding mode
$\bar{\varepsilon}$	Bound on the derivative estimation error in the sliding mode
α	Scalar multiplier
θ	time index

$\phi(\theta)$ Initial condition

$\mu(A)$ Matrix Measure of A

CHAPTER 5
SLIDING MODE CONTROL FOR UNCERTAIN INPUT DELAY SYSTEMS

5.1 Introduction

Time delay exists in various branches of engineering. From biological, optical, electric networks, to chemical reaction systems, time delay occurs and affects the stability and performance of a wide variety of systems [See Schell and Ross, 1986, Inamdar *et al.,* 1991]. Input delays occur as one of the common sources of time delay, such as in chemical processes, transportation lags and measurement delays, etc. It is well known that the existence of time delay degrades the control performances and makes the closed-loop stabilization difficult. Hence, the study of time delay systems has been paid considerable attention over the past years. The description of time-delay systems lead to differential-difference equations, which require the past values of the system variables. There are no known methods to get necessary and sufficient conditions for the robust stability of such systems. There are many approaches that yield sufficient conditions with varying degrees of sophistication.

The case of perfectly known (*i.e.,* no uncertainty) time-delay system is well studied. It is known that a stabilizable time-delay system [see Kamen *et al.,* 1985, Logemann, 1986] can always be stabilized by a finite dimensional controller. The study of state uncertain delay systems has been paid much attention. However, with regard to the control of uncertain input delay systems, only a few studies have been reported in the literature. Based on Riccati-equation approach, a robust controller is derived in Kojima and Ishjiama [1995]. Yan *et al.,* [1997] propose a functional observer and state feedback mechanism using the factorization approach. For a LQG/LTR method see Lee *et al.,* [1988]. Wu and

Chou [1996] propose a control algorithm from input-output feedback linearization by means of a parametrized coordinate transformation. Recently, an LMI based approach was given in Niculescu *et al.,* [1997] for delay-dependent closed loop stability of input-delay systems. However, the authors allowed uncertainty only in the input delay.

One of the common methods used in chemical engineering applications, for processes with time-delay, is to use the idea of the Smith-predictor scheme to cancel the effect of time delay [Stephanopoulos, 1984]. However, if the model is imperfect (uncertain) then Smith-predictor is known to give poor performance. Some approaches have been developed for time-delay compensation based on prediction strategies [See Henson and Seborg, 1994].

In this chapter we propose an approach based on the Sliding Mode Control [Hung *et al.,* 1993; Utkin, 1977, Decarlo *et al.,* 1988]. The sliding mode control approach possesses many advantages, *e.g.*, fast response, good transient performances, robustness to the variation in plant parameters and external disturbances. The authors in [Shyu and Yan, 1993; Oucheriah, 1995; Luo and Sen, 1993, Basker *et al.,* 1997] study the sliding mode control for uncertain systems with state delay. Also see Luo *et al.,* [1997] for a sliding mode approach for uncertain time delay systems with internal and external point delays via various types of feedback.

This chapter proposes a SMC controller to robustly stabilize an uncertain input-delay system with linear and possibly mismatching uncertainties. Sufficient robust stability conditions are derived. No matching conditions on uncertainties are assumed.

The rest of the chapter is organized as follows: In Section 5.2, a transformation is adapted to convert the original system with bounded input delay into one without any delay. The stabilization of the modified system implies the stability of the original system as long as the control law is bounded [see Kwon and Kim, 1980; Artstein, 1982]. Section 5.3 gives the guideline to design the sliding mode controller which consists of the switching function and the control law. In Section 5.4, the analysis of the perturbed system is carried out, and the reaching condition and the asymptotic stability of the sliding

mode are proven. In Section 5.5 some open issues in the sliding mode literature are raised and answers are provided. Section 5.6 considers constraints on the controller input. Section 5.7 gives an illustrative example with an open loop unstable plant. Conclusions and future work are discussed in Section 5.8.

5.2 Preliminaries and Problem Formulation

Consider the uncertain input delay system

$$\dot{x}(t) = Ax(t) + Bu(t) + B_d u(t-h) + f(x,t) \tag{5.1}$$

where $x(t) \in R^n$ is the state, $u(\cdot) \in R^m$ is the control, A, B and B_d are constant matrices with appropriate dimensions, h is a known constant time delay. The total plant uncertainty is bounded by

$$|f(x,t)| \le \beta |x(t)| \tag{5.2}$$

where $|.|$ denotes a vector-norm and $\|\|$ denotes the associated induced matrix-norm. Without loss of generality, the vector 2-norm $\|.\|_2$ is used. The positive scalar bound β is assumed to be known. In rest of the development, without loss of generality, we assume that $m < n$. Note that we do not assume any matching condition on the uncertainty. The method described here can be trivially extended to the case of multiple known delays.

Assumption 1: The pair $[A, B]$ is controllable and all the states are measurable.

Transformation into a delay free system

We consider a transformation to convert the original system (5.2) into an equivalent system without delay. This reduction enables the use of known stabilizing methods for linear systems without input delay. See Artstein [1982], for extensive theoretical discussions on the validity and the applications of this type of reduction and also see Fiagbedzi and Pearson [1986], for further generalizations.

Define the transformation

$$z(t) = x(t) + \int_{t-h}^{t} e^{A(t-h-\tau)} B_d u(\tau) d\tau \tag{5.3}$$

Assumption 2. The solution $z(t)$ is not affected by a change in $u(t)$ on a set of zero Lebesgue measure [Artstein, 1982].

For an arbitrary bounded control $u(t)$, the system (5.2) is transformed to (see Appendix D for details)

$$\dot{z}(t) = Az(t) + \overline{B}u(t) + f(x,t) \tag{5.4}$$

Where $\overline{B} = B + e^{-Ah} B_d$. It is assumed that $\overline{B}$ has full rank m. Then, via PBH rank test [Kailath, 1980] it is trivial to show that the system (5.4) is also controllable. Note that the system described by equation (5.4) is an uncertain linear system without input delay.

5.3 The Design of the Sliding Mode Controller

In the absence of perturbations, from (5.4), the nominal system is given by

$$\dot{z}(t) = Az(t) + \overline{B}u(t) \tag{5.5}$$

In the following we will design the switching function parameters on the basis of system (5.5). The design consists of two phases. The first phase is the construction of the switching function, so that, the nominal system restricted to the switching surface is asymptotically stable. The second phase involves the development of a control law, which must satisfy a sufficient condition for the existence and reachability of a sliding mode within a finite time period.

Design of the switching function

Since system in (5.5) is a typical linear system, choose the switching function to be

$$s(t) = Cz(t) \tag{5.6}$$

where C is an $m \times n$ matrix to be determined. The idea is to design the switching function so that the nominal system is asymptotically stable in the sliding mode [Woodham and

Zinober, 1993]. Note that other types of switching function such as an integral switching function can also be used to facilitate desirable switching dynamics. Since the matrix $\overline{B}$ has full rank m, it is possible to reduce $(n-m)$ of the rows to zero, *i.e.*, there exists a nonsingular matrix T such that

$$T\overline{B} = \begin{bmatrix} 0 \\ B_1 \end{bmatrix} \tag{5.7}$$

where B_1 is an $m \times m$ nonsingular matrix. In fact, we remark that such a T has a simple form for $m = 1$ and is given by

$$T = \begin{bmatrix} 1 & \cdots & 0 & -\left(\dfrac{\overline{B}(1)}{\overline{B}(n)}\right) \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & -\left(\dfrac{\overline{B}(n-1)}{\overline{B}(n)}\right) \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

The transformed states are given by

$$w(t) = Tz(t) \tag{5.8}$$

and (5.5) becomes

$$\dot{w}(t) = TAT^{-1}w(t) + T\overline{B}u(t) \tag{5.9}$$

while the switching function becomes

$$s(t) = CT^{-1}w(t) = Qw(t) \tag{5.10}$$

We introduce the following partitions for later use.

$$\begin{aligned} w(t) &= \begin{bmatrix} \overline{w}(t) \\ w_1(t) \end{bmatrix} \begin{matrix} (n-m) \\ (m) \\ (1) \end{matrix} & Q &= \underset{(n-m)}{[\overline{Q}} \quad \underset{(m)}{Q_1}](m) \\ T &= \underset{(n-m)}{[T_1} \quad \underset{(m)}{T_2}]\, n & TAT^{-1} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{matrix} (n-m) \\ (m) \\ (n-m) \quad (m) \end{matrix} \end{aligned} \tag{5.11}$$

equation (5.9) can be rewritten in the form

$$\dot{\overline{w}}(t) = A_{11}\overline{w}(t) + A_{12}w_I(t) \tag{5.11a}$$

$$\dot{w}_I(t) = A_{21}\overline{w}(t) + A_{22}w_I(t) + B_I u(t) \tag{5.11b}$$

Furthermore, (5.10) can be written in the form

$$s(t) = \overline{Q}\,\overline{w}(t) + Q_I w_I(t) \tag{5.12}$$

where $\overline{Q}$ is an $m \times (n-m)$ matrix, and Q_I is an $m \times m$ nonsingular matrix. Differentiating (5.12) and substituting (5.11) in to the dynamic equation yields

$$\dot{\overline{w}}(t) = \left(A_{11} - A_{12}Q_I^{-1}\overline{Q}\right)\overline{w}(t) + A_{I2}Q_I^{-1}s(t) \tag{5.13}$$

$$\dot{s}(t) = \left[\left(\overline{Q}A_{11} + Q_I A_{21}\right) - \left(\overline{Q}A_{12} + Q_I A_{22}\right)Q_I^{-1}\overline{Q}\right]\overline{w}(t) + \left(\overline{Q}A_{12} + Q_I A_{22}\right)Q_I^{-1}s(t) + Q_I B_I u(t) \tag{5.14}$$

When $s(t) \neq 0$, equations (5.13) and (5.14) describe the reaching mode. On the other hand, when the system is in sliding mode, *i.e.,* $s(t) = 0$ and $\dot{s}(t) = 0$, (5.13) yields

$$\dot{\overline{w}}(t) = \left(A_{11} - A_{12}Q_I^{-1}\overline{Q}\right)\overline{w}(t) \tag{5.15}$$

$$w_I(t) = -Q_I^{-1}\overline{Q}\,\overline{w}(t) \tag{5.15a}$$

Since the pair (A_{11}, A_{12}) is completely controllable (see the Appendix E for a proof), there exists an $m \times (n-m)$ matrix K that satisfies

$$\lambda(A_{11} - A_{12}K) = \lambda^o \tag{5.16}$$

where λ^o represents the desired poles in the sliding mode. Since A_{11} and A_{12} are known, a set of linear equations can be solved to yield K. Comparing (5.15) and (5.16), we have

$$\overline{Q} = Q_I K \tag{5.17}$$

Choosing $Q_I = I$ yields

$$Q = [\overline{Q} \quad Q_I] = [K \quad I] \tag{5.18}$$

Substituting (5.18) into (5.10) results in

$$C = QT = [K \;\; I_m]T \tag{5.19}$$

Therefore, the switching function has the form

$$s(t) = Cz(t) = [K \quad I]Tz(t) \tag{5.20}$$

It is instructive to recast the dynamics of the system in terms of our choice for C, *i.e.*, K. By using (5.11), it is easy to show the following:

$$\dot{\overline{w}}(t) = (A_{11} - A_{12}K)\overline{w}(t) + A_{12}s(t) \tag{5.21}$$

$$\dot{s}(t) = \left[K(A_{11} - A_{12}K) + (A_{21} - A_{22}K)\right]\overline{w}(t) + (KA_{12} + A_{22})s(t) + B_I u(t) \tag{5.22}$$

$$w_I(t) = s(t) - K\overline{w}(t) \tag{5.23}$$

We also remark that in the sliding mode, $w_I(t) = -K\overline{w}(t)$ and hence, the asymptotic stability of (5.15) ensures that $\lim_{t\to\infty} w_I(t) \;\to\; 0$.

Remarks:

(1) The above also illustrates an interesting system-theoretic point. We are seeking to control an *n*-dimensional system via a m-dimensional input. From (5.21)-(5.23), it is obvious that the input explicitly affects only a *m*-dimensional subset of this realization and the remaining *(n-m)* dimensional subsystem is affected via the feedback of the states as determined by the system. This information is not new. In fact, by recasting the system in the controllability realization, this fact can be trivially seen. But, in contrast to the controllability canonical form, (5.21)-(5.23) also guide in choosing a controller to exploit the fact that the matrix B_I is of rank *'m'*. More about this will follow in later sections.

(2) Note that with the choice of C, $(CB)^{-1} = {B_I}^{-1}$ and since B_I is nonsingular, $(CB)^{-1}$ always exists.

Design of the Control Law

The control law chosen is a combination of the traditional state feedback and a time varying discontinuous control element

$$u(t) = -(C\bar{B})^{-1}\left[CAz(t) + k|s(t)|^{\alpha}\,\mathrm{sgn}[s(t)] + d(t)\mathrm{sgn}[s(t)]\right] \qquad (5.24)$$

where the parameters are chosen such that $k > 0$, $0 \le \alpha \le 1.0$ and $d(t) = \beta|C\|x(t)| + \varepsilon$, with $\varepsilon > 0$. When $s(t)$ is small, a faster approach rate than the linear approach algorithm is realized. The state feedback is designed to cancel the effect of the state matrix A in the switching dynamics. Once the system is in the vicinity of the switching surface, the magnitude of the discontinuous control is linearly proportional to the norm of the states. Since the states $x(t)$ and $z(t)$ are available, realizing the above control law is straight forward. Using the partitions (5.11), it is also easy to verify that (5.24) is equivalent to

$$u = -(B_I)^{-1}\left\{\left[K(A_{11} - A_{12}K) + (A_{21} - A_{22}K)\right]\bar{w}(t) + (KA_{12} + A_{22})s(t)\right\}$$

$$+$$

$$-(B_I)^{-1}\left[k|s(t)|^{\alpha}\,\mathrm{sgn}[s(t)] + d(t)\mathrm{sgn}[s(t)]\right] \qquad (5.25)$$

Comparing (5.25) and (5.22), it is obvious that we exploit the fact B_I is non-singular $m \times m$ matrix, and obtain u by equating (5.22) to zero.

With the chosen controller, the switching dynamics (5.22) is of the form

$$\dot{s}(t) = -k|s(t)|^{\alpha}\,\mathrm{sgn}[s(t)] - d(t)\mathrm{sgn}[s(t)] + Cf(x(t),t) \qquad (5.26)$$

Remarks:

(i) From (5.26), it is obvious that the discontinuous control is used to overpower the effect of uncertainty. Since the only information we have about the uncertainty is the norm bound, we can not use any lower value of $d(t)$ and be guaranteed to reach the sliding mode within a finite time. From (5.25), it is clear that there will always be chattering for this choice of parameters.

(*ii*) For the nominal system $d(t)$ can be chosen arbitrarily close to zero.

(*iii*) The parameter α can be used in a time varying fashion to guarantee a faster rate of approach to the switching surface than the normal choice $\alpha = 1.0$. For example, $\alpha = 1.0$ for $|s(t)| \geq 1.0$ and $\alpha = 0.5$ for $|s(t)| = 1.0$ would assure a faster rate of approach than the proportion control (*i.e.*, $\alpha = 1.0$, $\forall t > 0$).

5.4 Analysis of the Perturbed System

In this section, the robust stability of the uncertain system (5.1) is analyzed. First, the analysis is carried out for system (5.4), in order to give conditions for reaching the sliding mode and the asymptotic stability in the sliding mode. Since system (5.4) is related to system (5.1) via (5.3), conditions are given under which robust stability can be inferred for the original system (5.1).

For an uncertain system to be robustly stable under sliding mode control, the following conditions have to be met: *(i)* The system should attain the *sliding mode* within a finite time; that is, the state trajectories should reach the switching surface and should stay on the switching surface for all time thereafter; and *(ii)* the system in the sliding mode should be asymptotically stable.

Reaching Condition for Sliding Mode.

Theorem 5.1: *The uncertain delay system (5.4) with model-uncertainty and bounded measurement uncertainty given by (5.2) achieves the desired sliding mode within a finite time if the control law (5.25) is applied .*

Furthermore, it can be shown that the value of α *chosen 1, then the sliding mode is attained within* $t = t_r$, *where*

$$t_r \leq \left(\frac{2}{k}\right) \ln\left[1 + \frac{k}{\varepsilon}\left(\sqrt{\frac{s^T(0)s(0)}{2}}\right)\right] \tag{5.27}$$

and the upper bound on the norm of the switching surface is given by

$$|s(t)| \le \left[\frac{1}{\sqrt{2}}|s(0)| + \frac{\varepsilon}{k}\right] e^{-\frac{k}{2}t} - \frac{\varepsilon}{k}. \tag{5.28}$$

Proof: We choose a positive definite Lyapunov function

$$V(t) = \frac{1}{2}\left(s^T(t)s(t)\right) \tag{5.29}$$

Our aim is to show that $\dot{V}(t) < 0$ for the choice of parameters k and ε. The development here is fairly standard.

$$\dot{V}(t) = s^T(t)\dot{s}(t)$$

utilizing (5.26) for $\dot{s}(t)$

$$= s^T(t)\left[k|s(t)|^{\alpha}\,\mathrm{sgn}[s(t)] - d(t)\mathrm{sgn}[s(t)] + Cf(x,t)\right] \tag{5.30}$$

From the available bounds on the uncertainty

$$|Cf(x,t)| \le |C||f(x,t)| \le \beta|C||x(t)| \le d \tag{5.31}$$

Using (5.31) in (5.30)

$$\dot{V}(t) \le -|s(t)|^{\alpha}\left[k s^T(t)\mathrm{sgn}[s(t)]\right] - d(t)\left[s^T(t)\mathrm{sgn}[s(t)]\right] + s^T(t)Cf(x,t) \tag{5.32}$$

We observe that

$$s^T(t)\mathrm{sgn}[s(t)] = \sum s_i{}^T(t)\mathrm{sgn}[s_i(t)]$$

$$= \sum |s_i(t)|[\mathrm{sgn}[s_i(t)]]^2$$

$$= \sum |s_i(t)| = |s(t)|_1 \tag{5.33}$$

Using (5.33) in (5.32)

$$\dot{V}(t) \le -k|s(t)|^{\alpha}|s(t)|_1 - \varepsilon|s(t)| \tag{5.34}$$

Since both the terms on the right hand side are negative, the system is guaranteed to attain the sliding mode in finite time. Furthermore, for the choice of $\alpha = 1.0$, (5.34) becomes

$$\dot{V}(t) \le -k|s(t)|^2 - \varepsilon|s(t)|_1 \tag{5.35}$$

since for a vector $|\cdot|_2 \le |\cdot|_1$.

Using (5.5.29)

$$\dot{V}(t) \le -kV(t) - \varepsilon V^{1/2}(t) \tag{5.36}$$

The above is a scalar difference equation, and from Appendix F,

$$\sqrt{V(t)} \le \left[\sqrt{V(0)} + \frac{\varepsilon}{\kappa}\right] e^{-\frac{k}{2}t} - \frac{\varepsilon}{k} \tag{5.37}$$

$$|s(t)| \le \left[|s(0)| + \frac{\sqrt{2}\varepsilon}{k}\right] e^{-\frac{k}{2}t} - \frac{\sqrt{2}\varepsilon}{k} \tag{5.38}$$

Equating the right hand side to zero we get the result. ∇

Corollary:

For the choice $0 \le \alpha < 1.0$, a tighter result can be obtained. Following the same steps as in the proof of the above theorem (see Appendix F), it can be shown that for a control law (5.24) with $d(t) = \beta|C\|x(t)|$, the sliding mode is achieved within $t = t_r$, where

$$t_r \le \frac{2}{k(1-\alpha)}\left[\frac{1}{2}s^T(0)s(0)\right]^{0.5(1-\alpha)}$$

and

$$|s(t)| \le \sqrt{2}\left[\frac{(1-\alpha)}{2}\left[\frac{2}{(1-\alpha)}[x(0)]^{0.5(1-\alpha)} - kt\right]\right]^{\left(\frac{2}{1-\alpha}\right)}, \quad 0 \le t \le t_r$$

This is advantageous because, we avoid an extra discontinuous control element, ε.

Asymptotic Stability in the Sliding Mode

Note that when the system is in the sliding mode, the control law reduces to a linear state feedback. In this chapter, we follow a Lyapunov-Razhamikhin type approach to derive sufficient stability condition.

The dynamic equation in sliding mode is given by

$$\dot{\bar{w}}(t) = (A_{11} - A_{12}K)\bar{w}(t) + T_1 f(x,t) \tag{5.39}$$

Since in the sliding mode, $w_1(t) = -K\bar{w}(t)$, it suffices to show the asymptotic stability of

(5.39). The first task is to convert the uncertainty description in terms of $\overline{w}(t)$.

$$z(t) = T^{-1}\begin{bmatrix}\overline{w}(t)\\ w_1(t)\end{bmatrix} = T^{-1}\begin{bmatrix}\overline{w}(t)\\ -K\overline{w}_1(t)\end{bmatrix} = T^{-1}\begin{bmatrix}I_{n-m}\\ -K\end{bmatrix}\overline{w}(t) \tag{5.40}$$

Next we seek an expression relating $x(t)$ and $z(t)$. From (5.5.3)

$$|x(t)| \le |z(t)| + \left|\int_{t-h}^{t} e^{A(t-h-\tau)} B_d u(\tau) d\tau\right| \tag{5.41}$$

where $u(t) = -(C\overline{B})^{-1}CAz(t)$.

Thus

$$|x(t)| \le |z(t)| + \left[\max_{-h\le\theta\le 0}\left\|e^{A\theta}\right\|\right] B_d (C\overline{B})^{-1}CA\left|\int_{t-h}^{t}|z(\tau)|d\tau\right| \tag{5.42}$$

The main hurdle at this point is to express $|z(\tau)|$ in terms of $|z(t)|$. To obviate this difficulty, we take Lyapunov--Razhamikhin type approach. First of all we show that for the case of matched uncertainties, proving asymptotic stability is straight forward.

Definition: *Matching condition* [Hung et al., 1993]. If the uncertainties satisfy

$$f(x,t) = \overline{B}\bar{f}(x,t)$$

where $\bar{f}(x,t) \in \Re^{m\times 1}$, then the matching condition is satisfied.

This means that the perturbation is in the image of $\overline{B}$.

$$Tf(x,t) = T\overline{B}\bar{f}(x,t)$$

$$= \begin{bmatrix}0\\ T_2 B_1 \bar{f}(x,t)\end{bmatrix}$$

Hence, in the sliding mode, the system equation becomes

$$\dot{\overline{w}}(t) = \left(A_{11} - A_{12}Q_1^{-1}\overline{Q}\right)\overline{w}(t) \tag{5.43}$$

Note that (5.43) is of the same form as the equation describing sliding mode for the nominal system (5.15). This implies that the uncertainties will have no effect on the system in the sliding mode. This is one of the major advantages of the sliding mode design. The state feedback is simply used to place the poles of the nominal system in the

sliding mode. However, the assumption of matched uncertainties is a highly restrictive one, as for example in the $m = 1$ case, we are constraining the perturbation to be a scaled version of $\overline{B}$. One should note that the situation represented in (5.43) is an ideal case. That is, if the controller is able to exactly cancel the perturbation, then the sliding mode is independent of the perturbation. This obviously suggests that if one could estimate and cancel the perturbation (or) the effect of the perturbation on the sliding mode, then the system in sliding mode can be made to be independent of the matching perturbations. These kinds of approaches are commonly called *perturbation compensation* and have been applied to linear state delay systems [Chan et al., 1996]. Even if mismatched uncertainties are present, *perturbation compensation* can be employed to reduce the magnitude of the required discontinuous control.

In the mismatched case, it is not possible to completely cancel the perturbation and there will be a residual effect of the perturbations in the sliding mode and the state feedback should be such that the overall system is asymptotically stable. Theorem 5.2 gives a sufficient condition for the mismatched case.

Before we state and prove Theorem 5.2, we explain about a difficulty that commonly arises in proving asymptotic stability in state-delayed systems and input-delayed systems (as the control usually involves state feedback, the input-delay problem has relevance and resemblance to state-delayed problem). There are two classical approaches in the literature for proving asymptotic stability for delayed systems. Over the years there have been many incremental improvements, but the message is that all methods are conservative. This is because the differential equations that determine the dynamics of the system have two difficult elements. One, the perturbations. Two, the delayed states. We know only the norm bound on the perturbation, and while using this sufficiency creeps in. The problem with delayed states is more interesting. In using Lyapunov-Razhamikhin type approaches, a restriction is imposed on the dynamics of the system.

Consider a Lyapunov candidate

$$V(t) = \overline{w}^T(t) P \overline{w}(t)$$

where $P = P^T > 0$.

At this point, we introduce the usual Razhamikhin type restriction on the system. We restrict that

$$V(\theta) \le \delta V(t), \quad t - h \le \theta \le t, \forall \delta > 1$$

It is important to understand, how restrictive the above assumption is. The reason for using such an assumption is that it allows one to bound the norm of the delayed states of the system in terms of the norm of the present states of the system. Obviously any condition based on this will only be sufficient. Since the following condition holds for any positive-definite matrix $P = P^T > 0$, and $\overline{w} \in \Re^{n-m}$

$$\lambda_{\min}(P)\overline{w}^T\overline{w} \le \overline{w}^T P \overline{w} \le \lambda_{\min}(P)\overline{w}^T\overline{w}$$

it follows that

$$|\overline{w}(\theta)| \le \delta\alpha|\overline{w}(t)|$$

where $\alpha = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}$,

The above condition is not very restrictive as it is true for some $\delta_1 > \delta > 1$, for all stable systems whose state norm continuously decreases (and it is easy to show that for unstable systems, the above is true for any positive δ). The actual values of δ and α (and hence P) quantify the decay rate of the system. This condition, however could be violated for a short durations, for example, when a delayed disturbance hits the system (*i.e,* the norm of the states could increase). However, the controller will act against the disturbance and restore the performance. Hence, temporary violation of this constraint is not fatal, and will not affect the stability conclusion.

Theorem 5.2: *The system (5.4) in sliding mode with the mismatched uncertainties (5.2) is asymptotically stable if the following condition is satisfied*

$$\beta\left\|T^{-1}\begin{bmatrix} I_{n-m} \\ -K \end{bmatrix}\right\|\left\{1+h\alpha\left[\max_{-h\le\theta\le 0}\left\|e^{A\theta}\right\|\left\|B_d(C\overline{B})^{-1}CA\right\|\right]\right\} < \frac{\lambda_{\min}(Q)}{2\|P\|\|T_1\|} \qquad (5.44)$$

where

$$\alpha = \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \text{ and } P>0 \text{ is the solution of}$$

$$P(A_{11}-A_{12}K)+(A_{11}-A_{12}K)^T P = -Q,\ Q>0 \qquad (5.45)$$

Proof. Under the similarity transformation (5.8) the system (5.4) transforms to (5.11). If the uncertainties are present, using (5.11) equation (5.11a)-(5.11b) adopts the form

$$\dot{\overline{w}}(t) = A_{11}\overline{w}(t) + A_{12}w_1(t) + T_1 f(x,t) \qquad (5.46)$$

$$\dot{w}_1(t) = A_{21}\overline{w}(t) + A_{22}w_1(t) + B_1 u(t) + T_2 f(x,t) \qquad (5.47)$$

Consider a Lyapunov candidate

$$V(t) = \overline{w}^T(t)P\overline{w}(t) \qquad (5.48)$$

where $P = P^T > 0,\ P = P^T > 0, P \in \Re^{(n-m)\times(n-m)}$.

Taking the derivative of (5.48)

$$\dot{V}(t) = \overline{w}^T(t)\left[P(A_{11}-A_{12}K)+(A_{11}-A_{12}K)^T P\right]w^T(t) + 2w^T(t)PT_1 f(x,t) \qquad (5.49)$$

At this point, we make the usual Razhamikhin type restriction on the system. We restrict that

$$V(\theta) \le \delta V(t),\ t-h \le \theta \le t, \forall \delta > 1 \qquad (5.50)$$

it follows that,

$$|\overline{w}(\theta)| \le \delta \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} |\overline{w}(t)| \tag{5.51}$$

Denoting $\alpha = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}$, using this in (5.42)

$$|x(t)| \le \left\| T^{-1} \begin{bmatrix} I_{n-m} \\ -K \end{bmatrix} \right\| \left\{ 1 + h\delta\alpha \left[\max_{-h \le \theta \le 0} \left\| e^{A\theta} \right\| \left\| B_d (C\overline{B})^{-1} CA \right\| \right] \right\} |\overline{w}(t)| \tag{5.52}$$

Therefore,

$$|f(x,t)| \le \beta \left\| T^{-1} \begin{bmatrix} I_{n-m} \\ -K \end{bmatrix} \right\| \left\{ 1 + h\delta\alpha \left[\max_{-h \le \theta \le 0} \left\| e^{A\theta} \right\| \left\| B_d (C\overline{B})^{-1} CA \right\| \right] \right\} |\overline{w}(t)| \tag{5.53}$$

We combine all the terms to get

$$\beta_1 = \beta \left\| T^{-1} \begin{bmatrix} I_{n-m} \\ -K \end{bmatrix} \right\| \left\{ 1 + h\delta\alpha \left[\max_{-h \le \theta \le 0} \left\| e^{A\theta} \right\| \left\| B_d (C\overline{B})^{-1} CA \right\| \right] \right\}$$

$$|f(x,t)| \le \beta_1 |\overline{w}(t)| \tag{5.54}$$

hence

$$\dot{V}(t) = \overline{w}^T(t) \left[P(A_{11} - A_{12}K) + (A_{11} - A_{12}K)^T P \right] \overline{w}(t) + 2\beta_1 \|P\| \|T_1\| |\overline{w}(t)|^2 \tag{5.55}$$

Choose $P > 0$, $Q > 0$, such that

$$P(A_{11} - A_{12}K) + (A_{11} - A_{12}K)^T P = -Q$$

$$\le -\overline{w}^T(t) Q \overline{w}(t) + 2\beta_1 \|P\| \|T_1\| |\overline{w}(t)|^2 \tag{5.56}$$

$$\dot{V}(t) \le \left[-\lambda_{\min}(Q) + 2\beta_1 \|P\| \|T_1\| \right] |\overline{w}(t)|^2 \tag{5.57}$$

Hence a sufficient condition for asymptotic stability in the sliding mode is given by

$$\beta_1 < \frac{\lambda_{\min}(Q)}{2\|P\| \|T_1\|}$$

i.e.,

$$\beta \left\| T^{-1} \begin{bmatrix} I_{n-m} \\ -K \end{bmatrix} \right\| \left\{ 1 + h\delta\alpha \left[\max_{-h \le \theta \le 0} \left\| e^{A\theta} \right\| \left\| B_d (C\overline{B})^{-1} CA \right\| \right] \right\} < \frac{\lambda_{\min}(Q)}{2\|P\| \|T_1\|} \tag{5.58}$$

Note that if the above expression is true for $\delta = 1$ then there exists a $\delta > 1$, sufficiently small that it will still hold true. Therefore, a sufficient condition for stability is given by the above expression, with $\delta = 1$. ∇

Remarks:

(i) In the sliding mode, the derivative of the Lyapunov function is given

$$\dot{V}(t) \le \left[-\lambda_{\min}(Q) + 2\beta_1 \|P\| \|T_1\|\right] |\overline{w}(t)|^2 < 0.$$

Denoting $\tilde{\varepsilon} := -\left(-\lambda_{\min}(Q) + 2\beta_1 \|P\| \|T_1\|\right)$, we can obtain

$$\begin{aligned} \dot{V}(t) &\le -\tilde{\varepsilon} |\overline{w}(t)|^2 \\ &\le -\left[\frac{\tilde{\varepsilon}}{\lambda_{\min}(P)}\right] V(t) \end{aligned} \tag{5.59}$$

This explains why we can get only asymptotic stability guarantee, that is, we can only state that the norm of the states will reach zero as time goes to infinity. In the next section we will elaborate more about this issue.

(ii) For a given system description, once the poles of the closed-loop nominal system are chosen, everything except P and hence Q is known. Hence P is an optimization parameter. We remark that the best choice is to make sure that the condition number of P is minimized while satisfying (5.44)-(5.45). For large systems, this constrained optimization problem can be cast as an convex LMI problem and easily solved through available commercial software [Niculescu *et al.,* 1997].

Theorem 5.3. *If the uncertain system (5.4) is asymptotically stable, it implies the asymptotic stability of system (5.1).*

Proof. From the transformation (5.3), we have

$$|x(t)| \le |z(t)| + h\left[\max_{-h \le \theta \le 0} \left|e^{A\theta}\right|\right] \cdot |B_d| \cdot |u_t| \tag{5.60}$$

From Theorem 5.2, the transformed system (5.4) is asymptotically stable. Hence,

$\lim_{t\to\infty} |z(t)| \to 0$. Since $u(t) = -(CB)^{-1}CAz(t)$, in the sliding mode, $\lim_{t\to\infty} |u(t)| \to 0$ From (5.60) the result follows. ∇

Implementation Issues

Due to inherent limitations such as the switching delay, system inertia, etc., the sliding motion does not take place in any real system. Instead, the trajectories chatter in a neighborhood of the switching surface. The chattering phenomenon is undesirable because it may adversely affect the actuating mechanism and excite the high-frequency unmodeled plant dynamics, leading to instability. The discussion about chattering and the ways to alleviate this problem, as outlined in Chapter 4, applies to the present case of stabilizing systems with input-delay as well.

5.5 Some Open Issues in Sliding Mode Literature

Although the sliding mode control approaches have been used for a long time for a variety of systems, there are nevertheless some open issues. We raise the following issues of interest:

(1) Is it possible to guarantee that the norm of the states continuously decrease during the reaching phase?. If so, what is the non-trivial upper bound, on the norm of the states?

(2) Is it possible to get an upper bound the norm of the states when the system is in the reaching phase (other than of course, the trivial estimate $|x(0)|$) ?

(3) We are able to prove that the sliding mode will be attained within a finite time. At the same time, the best we can prove for the system in the sliding mode is asymptotic stability. We also seek to explain why this is so.

We also remark that these questions apply to all sliding mode control schemes (for linear uncertain, Uncertain linear systems with state delay and so on..) and not just for the input-delay case that we consider here. Note that the usual approach in the literature for

sliding mode systems is to prove the reaching condition within a finite time (as we did via Theorem 5.1), and then prove asymptotic stability for the reduced system in the sliding mode (as in Theorem 5.2). However, there are problems with this approach. For example, one does not have an estimate of the norm of the states, when the system reaches the sliding mode. Hence, we do not know the proper initial condition for the scalar Lyapunov differential equation given by (5.59). One only hopes that it is less than the norm corresponding to the initial condition. Even assuming so, it is still highly conservative. What we need is a way of estimating a tight upper-bound on the norm of the states, during the reaching phase. If we obtain such an expression, then we will be able to answer questions (Q1)-(Q2).

Before we seek to get an analytical expression to the above questions, it is instructive to ponder over the significance of Theorem 5.1. What (5.27)-(5.28) imply is that the norm of a linear combination of the states continuously decreases and attains zero within a finite time. But, a linear combination of two time-dependent functions (here states of the system) may be decreasing towards zero, while the individual elements may be diverging in opposite direction, making their sum (or difference) approach zero. Hence, it seems like that we do not even have a guarantee that the norm of the states of the system will continuously decrease, when the norm of the switching function continuously decreases and attains the sliding mode. In other words, the central question here is, can one obtain sliding mode when the system is unstable .

However, upon reflection, one simple fact emerges. Consider the linear uncertain system (5.4) and bounded perturbation (5.2). The solution will be driven by the interaction of the control and perturbations and the rate at which the states converge (or) diverge will depend on the state transition matrix, and whether or not it is stable enough to overcome the perturbations. Ultimately the form of the solution will contain complex exponentials, apart from other terms. Since, the perturbation is bound by the norm of the states, an unique solution does exist. It is not our intention to derive it. In fact, it is impossible to

derive it, unless the form of the perturbation is exactly known. Now, Theorem 5.1 implies that the norm of a linear combination of the complex exponential terms is continuously decreasing. If the system is unstable, it is the real part of the complex exponentials that will dominate the system dynamics. In that case, the resulting linear combination of the complex exponentials will diverge. Hence, it is not possible for $|s(t)|$ to continuously decrease and attain zero within a finite time, if the system is unstable.

This however, does not necessarily mean that the norm of the states will continuously decrease. For example, for a system with state delay, when the delayed states first come into action, there could be a temporary increase in the norm. Alternately, there could be a non-linear uncertainty that satisfies all our constraints and still has a spike leading to an increase in the norm of the states. Of course, the controller will seek to react to this situation and bring the system under control. Therefore, although mathematically not precise, it is nevertheless acceptable to assume in general, that the norm of the states decrease in the reaching phase. This conclusion is important to assess the behavior of the state norm in the reaching phase. For example, without such a guarantee, it is impossible to bound the discontinuous control signal $d(t)$. From (5.24), $d(t) = \beta|C\|x(t)| + \varepsilon$. Hence, the value of the discontinuous control is linked to the norm of the states. In proving asymptotic stability in the sliding mode, we obviate this difficulty, as $\text{sgn}(s(t)) = 0$. With the above conclusion, however, one could make a confident assumption on the expected value of $d(t)$, much in the same vein of the Razhamikhin type assumption, that seeks to restrict the norm of the states.

The following lemma is needed for the ensuing developments.

Lemma 5.1

Consider the following scalar differential equation

$$\dot{x}(t) \le -ax(t) + \left(b\exp(-ct/2) + e\ \exp(-\delta_2 t/2) - d\right)x^{1/2}(t) \tag{5.61}$$

with the initial condition $x(0)$ *and the following constraints:*

(1) The constants $(a,b,c,d,e) > 0$ *and* $a \neq c$ *and* $a \neq \delta_2$

(2) $x > 0,\ \forall t\ > 0$ *and* $\sqrt{x} > 0,\ \forall t\ > 0$

Then,

$$\sqrt{x(t)} \le \left[\sqrt{x(0)} + \frac{b}{a}\right]\exp\left(-\frac{a}{2}t\right) + \frac{b}{(a-c)}\left[\exp\left(-\frac{c}{2}t\right) - \exp\left(-\frac{a}{2}t\right)\right] + \frac{e}{(a-\delta_2)}\left[\exp\left(-\frac{\delta_2}{2}t\right) - \exp\left(-\frac{a}{2}t\right)\right] - \frac{d}{a} \tag{5.62}$$

Proof: See Appendix G ∇

Remark:

In (5.62) the first three terms are positive, while the third term is a negative constant. Furthermore, the first term continuously decreases from the initial value, while the second and terms always have a maxima. As a parallel we remark that the second and third terms in (5.62) look like the response of an under damped second order system to unit impulse input.

Upper Bound on the Norm of the States During the Reaching Phase

In this section, the aim is to find an upper bound on the norm of the states, for the stable closed-loop system. The analysis for the nominal system is straightforward and is presented in the sequel. However for the perturbed system, the analysis is quite involved. In the reaching phase, the dynamic equations are given by

$$\dot{\overline{w}}(t) = (A_{11} - A_{12}K)\overline{w}(t) + T_1 f(x,t) + A_{12}s(t) \tag{5.63}$$

$$w_1(t) = s(t) - K\overline{w}(t) \tag{5.64}$$

In complete parallel to the proof of Theorem 5.2, consider a Lyapunov candidate

$$V(t) = \overline{w}^T(t)\hat{P}\overline{w}(t) \tag{5.65}$$

where $\hat{P} = \hat{P}^T > 0$, $\hat{P} = \hat{P}^T > 0$, $\hat{P} \in \Re^{(n-m)\times(n-m)}$ and impose the restriction

$$V(\theta) \le \delta V(t),\ \ t-h \le \theta \le t,\ \forall\, \delta > 1 \tag{5.66a}$$

It is straightforward to show from (5.66a) that

$$|\overline{w}(\theta)| \le \delta \left(\frac{\lambda_{\max}(\hat{P})}{\lambda_{\min}(\hat{P})} \right) |\overline{w}(t)| \tag{5.66b}$$

Choose $\hat{P} > 0, \quad Q > 0$, such that

$$\hat{P}(A_{11} - A_{12}K) + (A_{11} - A_{12}K)^T \hat{P} = -Q \tag{5.67}$$

Taking the derivative of (5.65) yields

$$\begin{aligned} \dot{V}(t) = \overline{w}^T(t) \left[\hat{P}(A_{11} - A_{12}K) + (A_{11} - A_{12}K)^T \hat{P} \right] w^T(t) \\ + 2w^T(t)\hat{P}T_1 f(x,t) + 2w^T(t)\hat{P}A_{12}s(t) \end{aligned} \tag{5.68}$$

Converting the Uncertainty in Terms of $\overline{w}(t)$

Again, the challenge is to convert the uncertainty description in terms of $\overline{w}(t)$. Since $|f(x(t),t)| \le \beta |x(t)|$, we run into a problem, since the stated aim is to find an upper bound on $|x(t)|$. In the ensuing development, we will pinpoint this difficulty and propose an additional assumption on the perturbation that will enable us to obviate this hurdle.. From Theorem 5.1, we have

$$|s(t)| \le \left[|s(0)| + \frac{\sqrt{2}\varepsilon}{k} \right] e^{-\frac{k}{2}t} - \frac{\sqrt{2}\varepsilon}{k} \tag{5.69}$$

For simplicity, we denote

$$\hat{b} = \left[|s(0)| + \frac{\sqrt{2}\varepsilon}{k} \right], \quad \hat{c} = k, \quad \hat{a} = \frac{\sqrt{2}\varepsilon}{k}.$$

Hence $|s(t)| \le \hat{b} e^{-\frac{\hat{c}}{2}t} - \hat{a}$.

In the reaching mode,

$$z(t) = T^{-1}w(t) = T^{-1} \begin{bmatrix} \overline{w}(t) \\ w_1(t) \end{bmatrix} = T^{-1} \begin{bmatrix} \overline{w}(t) \\ s(t) - K\overline{w}(t) \end{bmatrix}$$

$$z(t) = T^{-1} \begin{bmatrix} I_{n-m} \\ -K \end{bmatrix} \overline{w}(t) + T^{-1} \begin{bmatrix} 0_{n-m} \\ s(t) \end{bmatrix}$$

$$|z(t)| \le \left\| T^{-1} \begin{bmatrix} I_{n-m} \\ -K \end{bmatrix} \right\| |\overline{w}(t)| + \left\| \hat{T}_2 \right\| |s(t)|$$

where $\hat{T}_2$ is a matrix of dimension $n \times m$, partitioned out of T^{-1}.

Since we used the transformation (5.3), from (5.42)

$$|x(t)| \le |z(t)| + \left| \int_{t-h}^{t} e^{A(t-h-\tau)} B_d u(\tau) d\tau \right|$$

where the controller is given by

$$u(t) = -\left(C\overline{B}\right)^{-1} \left[CAz(t) + k|s(t)|\text{sgn}[s(t)] + d(t)\text{sgn}[s(t)] \right]$$

since $\|\text{sgn}[s(t)]\| \le \sqrt{m}$, we take an upper bound

$$|u(t)| \le \left| \left(C\overline{B}\right)^{-1} CAz(t) \right| + \sqrt{m} \left\| \left(C\overline{B}\right)^{-1} \right\| \left[k\hat{b}e^{-\frac{\hat{c}}{2}t} - k\hat{d} + d(t) \right]$$

$$|x(t)| \le |z(t)| + \left[\max_{-h \le \theta \le 0} \left\| e^{A\theta} \right\| \right] \left\| B_d (C\overline{B})^{-1} CA \right\| \int_{t-h}^{t} |z(\tau)| d\tau$$

$$+ \sqrt{m} \left[\max_{-h \le \theta \le 0} \left\| e^{A\theta} \right\| \right] \left\| B_d \right\| \left\| (C\overline{B})^{-1} \right\| \int_{t-h}^{t} \left[k\hat{b}e^{-\frac{\hat{c}}{2}\tau} - k\hat{d} + d(\tau) \right] d\tau$$

Substituting for $|z(t)|$ in terms of $|\overline{w}(t)|$ and $|s(t)|$, and using Razhamikhin type assumption for the delayed states, via (5.66b), results in

$$\begin{aligned} |x(t)| \le & \left\| T^{-1} \begin{bmatrix} I_{n-m} \\ -K \end{bmatrix} \right\| |\overline{w}(t)| + \left\| \hat{T}_2 \right\| \left(\hat{b}e^{-\frac{\hat{c}}{2}\tau} - \hat{d} \right) \\ & + h\delta\alpha \left\| T^{-1} \begin{bmatrix} I_{n-m} \\ -K \end{bmatrix} \right\| \max_{-h \le \theta \le 0} \left\| e^{A\theta} \right\| \left\| B_d (C\overline{B})^{-1} CA \right\| |\overline{w}(t)| \\ & + \left\| \hat{T}_2 \right\| \max_{-h \le \theta \le 0} \left\| e^{A\theta} \right\| \left\| B_d (C\overline{B})^{-1} CA \right\| \int_{t-h}^{t} \left(\hat{b}e^{-\frac{\hat{c}}{2}\tau} - \hat{d} \right) d\tau \qquad (5.70a) \\ & + \sqrt{m} \left[\max_{-h \le \theta \le 0} \left\| e^{A\theta} \right\| \right] \left\| B_d \right\| \left\| (C\overline{B})^{-1} \right\| \int_{t-h}^{t} \left[k\hat{b}e^{-\frac{\hat{c}}{2}\tau} - k\hat{d} + d(\tau) \right] d\tau \end{aligned}$$

In the above equation, the only unknown quantity is $d(t)$. According to our control law, $d(t) = \beta \|C\| |x(t)| + \varepsilon$, and our stated aim is to find an upper bound on the norm of the

states (note that this difficulty does not arise for the nominal system). This difficulty can be overcome, if an additional assumption can be made on the uncertainty description. We assume, that the uncertainty vector $|f(\boldsymbol{x}(t),t)|$ has finite energy. Hence, it can be bounded by a decaying exponential and can be represented as $d(t) = \hat{a}_3 \exp(-\delta_2 t),\ \delta_2 > 0$. It is assumed that the actual value of δ_2 is known.

To simplify notation, we denote

$$\tilde{a}_1 := \left\|\hat{T}_2\right\| \max_{-h\le\theta\le 0}\left\|e^{A\theta}\right\| \left\|B_d(C\overline{B})^{-1}CA\right\|,$$

$$\tilde{a}_2 := k\sqrt{m}\left[\max_{-h\le\theta\le 0}\left\|e^{A\theta}\right\|\right] \left\|B_d\right\| \left\|(C\overline{B})^{-1}\right\|$$

$$\tilde{a}_3 := (\tilde{a}_1 + \tilde{a}_2)\beta\hat{d}h$$

$$|f(\boldsymbol{x},t)| \le \beta|x(t)|$$

$$|f(\boldsymbol{x},t)| \le \beta_1|\overline{w}(t)| + \beta(\tilde{a}_1+\tilde{a}_2)\left(\frac{2\hat{b}}{\hat{c}}\exp\left(\frac{\hat{c}h}{2}\right)-1\right)\exp\left(\frac{-\hat{c}t}{2}\right) - \tilde{a}_3$$

$$+ \beta\left(\left\|\hat{T}_2\right\|\right)\left(\hat{b}\exp\left(\frac{-\hat{c}t}{2}\right)-\hat{d}\right) + \beta\left(\frac{\hat{a}_3}{\delta_2}\right)\left(\exp(\delta_2 h)-1\right)\exp(-\delta_2 t)$$

where $\beta_1 = \beta\left\|T^{-1}\begin{bmatrix} I_{n-m} \\ -K \end{bmatrix}\right\|\left\{1 + h\delta\alpha\left[\max_{-h\le\theta\le 0}\left\|e^{A\theta}\right\|\right]\left\|B_d(C\overline{B})^{-1}CA\right\|\right\}$

To further simplify notation, we denote

$$\tilde{m} = \beta(\tilde{a}_1+\tilde{a}_2)\left(\frac{2\hat{b}}{\hat{c}}\exp\left(\frac{\hat{c}h}{2}\right)-1\right) + \beta\hat{b}\left\|\hat{T}_2\right\|$$

$$\tilde{n} = \beta\left(\frac{\hat{a}_3}{\delta_2}\right)\left(\exp(\delta_2 h)-1\right)$$

$$\tilde{p} = \tilde{a}_3 + \beta\hat{d}\left\|\hat{T}_2\right\|$$

$$|f(\boldsymbol{x},t)| \le \beta_1|\overline{w}(t)| + \tilde{m}\exp\left(\frac{-\hat{c}t}{2}\right) - \tilde{p} + \tilde{n}\exp(-\delta_2 t) \qquad (5.70b)$$

From (5.67), (5.68) and (5.70b) we have

$$\dot{V}(t) < -\overline{w}^T(t)Q\overline{w}(t) + 2|w(t)|\left\|\hat{P}\right\|\left(\|T\|\,|f(x,t)| + \|A_{12}\|\,|s(t)|\right) \quad (5.71a)$$

$$\begin{aligned}\dot{V}(t) \;<\; & \left(-\lambda_{\min}(Q) + 2\beta_1|w(t)|\left\|\hat{P}\right\|\|T_1\|\right)|\overline{w}(t)|^2 + 2|w(t)\|\hat{P}\|\,\|A_{12}\|\left(\hat{b}\exp\left(\frac{-\hat{c}t}{2}\right) - \hat{d}\right) \\ & + 2|w(t)|\left\|\hat{P}\right\|\|T_1\|\left(\tilde{m}\exp\left(\frac{-\hat{c}t}{2}\right) - \tilde{p} + \tilde{n}\exp(-\delta_2 t)\right)\end{aligned} \quad (5.71b)$$

Denoting

$$a = -\left(-\lambda_{\min}(Q) + 2\beta_1|w(t)|\left\|\hat{P}\right\|\|T_1\|\right)\Big/\lambda_{\max}\left(\hat{P}\right)$$

$$b = 2\left[\|T_1\|\,\tilde{m} + \|A_{12}\|\hat{b}\right]\left(\frac{\lambda_{\max}\left(\hat{P}\right)}{\sqrt{\lambda_{\min}\left(\hat{P}\right)}}\right)$$

$$d = 2\|T_1\|\tilde{p}\left(\frac{\lambda_{\max}\left(\hat{P}\right)}{\sqrt{\lambda_{\min}\left(\hat{P}\right)}}\right)\|T_1\|$$

$$e = 2\|T_1\|\tilde{n}\left(\frac{\lambda_{\max}\left(\hat{P}\right)}{\sqrt{\lambda_{\min}\left(\hat{P}\right)}}\right)$$

equation (5.71b) can be written as

$$\begin{aligned}\dot{V}(t) \;<\; & -a\lambda_{\max}\left(\hat{P}\right)|\overline{w}(t)|^2 + \left(b\exp\left(\frac{-\hat{c}t}{2}\right) - d\right)|w(t)|\sqrt{\lambda_{\min}\left(\hat{P}\right)} \\ & + \left(e\exp\left(\frac{-\delta_2 t}{2}\right)\right)|w(t)|\sqrt{\lambda_{\min}\left(\hat{P}\right)}\end{aligned}$$

$$\dot{V}(t) \;<\; -aV + \left(b\exp\left(\frac{-\hat{c}t}{2}\right) + e\;\exp\left(\frac{-\delta_2 t}{2}\right) - d\right)\sqrt{V} \quad (5.72)$$

We remark that (5.72) is in a form similar to (5.61) and hence it is straightforward to get a bound on the norm of the states during the reaching phase.

Implication of the Result

Using Lemma 5.1, for the system (5.72), the norm of the states can be bounded. The maximum value of the upper bound and the time at which it occurs can also be easily computed, since the constants are known. This explicitly answers questions (1) and (2).

Upper Bound on the Norm of the States for the Nominal System

For the nominal system, from (5.71a), it follows that the following equation holds.

$$\dot{V}(t) < -\overline{w}^T(t)Q\overline{w}(t) + 2|w(t)|\,\left|\hat{P}\right|\,|A_{12}|\,|s(t)|$$

Hence,

$$\dot{V}(t) < -\lambda_{\min}(Q)\,|\overline{w}(t)|^2 + 2|w(t)|\,\left|\hat{P}\right|\,|A_{12}|\left(\hat{b}\exp\left(\frac{-\hat{c}t}{2}\right) - \hat{d}\right)$$

$$\dot{V}(t) < -\left(\frac{\lambda_{\min}(Q)}{\lambda_{\max}\left(\hat{P}\right)}\right)\lambda_{\max}\left(\hat{P}\right)|\overline{w}(t)|^2 + \left(\frac{2\,\left|\hat{P}\right|\,|A_{12}|}{\sqrt{\lambda_{\min}\left(\hat{P}\right)}}\right)\left(\hat{b}\exp\left(\frac{-\hat{c}t}{2}\right) - \hat{d}\right)\sqrt{\lambda_{\min}\left(\hat{P}\right)}\,|w(t)|$$

$$\dot{V}(t) < -\left(\frac{\lambda_{\min}(Q)}{\lambda_{\max}\left(\hat{P}\right)}\right)V(t) + \left(\frac{2\,\left|\hat{P}\right|\,|A_{12}|}{\sqrt{\lambda_{\min}\left(\hat{P}\right)}}\right)\left(\hat{b}\exp\left(\frac{-\hat{c}t}{2}\right) - \hat{d}\right)\sqrt{V(t)}$$

Then, from Lemma 5.1, it follows that an upper bound on the norm of the states can be established for the nominal system, without resorting to any additional approximations.

Caveat

It is easy to show that the solution of (5.72) goes to zero within a finite time. Does that mean that we are proving that the norm of the states are guaranteed to go to zero within a finite time. The temptation to reach such a conclusion must be avoided. This is because, (5.72) is valid only until the time sliding mode is attained. After that it is the differential equation (5.39) and the stability condition (5.58) that hold true.

Why only Asymptotic Stability Guarantees?

Now we turn our attention to the third issue that we raised. Why is it not that we are able to prove that we can drive a m dimensional manifold of the system to origin within a finite time, whereas it is impossible to get such a result for the remaining $(n-m)$ dimensional manifold of the system. The answer lies in the fact that we are limited by the rank of the matrix $\boldsymbol{B}$. The following theorem is a step towards understanding the inherent system limitation.

Theorem 5.4. *Consider the following nominal system*

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t),$$

(1) If the control matrix $\boldsymbol{B}$ is of full rank, the system trajectories can be driven to the origin within a finite time by the discontinuous control law

$$\boldsymbol{u}(t) = -(\boldsymbol{B})^{-1}\left[\boldsymbol{A}\boldsymbol{x}(t) + \alpha_1 \boldsymbol{x}(t) + \alpha_2 \operatorname{sgn}(\boldsymbol{x}(t))\right],\ \alpha_1 > 0,\ \alpha_2 > 0$$

(2) If the control matrix $\boldsymbol{B}$ is not of full rank, then it is not possible to drive the system trajectories to the origin within a finite time by a discontinuous control law.

Proof: First consider the case, where $\boldsymbol{B}$ is of full rank. Then, applying the above discontinuous control law results in

$$\dot{\boldsymbol{x}}(t) = -\alpha_1 \boldsymbol{x}(t) - \alpha_2 \operatorname{sgn}(\boldsymbol{x}(t))$$

Introduce a simple Lyapunov function of the form

$$V(t) = \frac{1}{2}\boldsymbol{x}^T(t)\boldsymbol{x}(t)$$

$$\begin{aligned}\dot{V}(t) &= \boldsymbol{x}^T(t)\dot{\boldsymbol{x}}(t) \\ &= -2\alpha_1 V(t) - \alpha_2 \boldsymbol{x}^T(t)\operatorname{sgn}(\boldsymbol{x}(t))\end{aligned}$$

Since $\boldsymbol{x}^T(t)\operatorname{sgn}(\boldsymbol{x}(t)) = |\boldsymbol{x}(t)|_1$, the above equation can be rewritten as

$$\dot{V}(t) < -2\alpha_1 V(t) - \sqrt{2}\alpha_2 \sqrt{V(t)}$$

Following Appendix F, it is straightforward to show that the trajectories of the system will

origin within a finite time. Furthermore, it is also possible to get an explicit upper bound for the norm of the states. This completes the first part of the proof.

Now consider the case where $\boldsymbol{B}$ is not full rank. Consider the discontinuous control law

$$u(t) = -K_1 x(t) - K_2 \operatorname{sgn}(x(t))$$

The system dynamics is given by

$$\dot{x}(t) = \overline{A}x(t) - BK_2 \operatorname{sgn}(x(t)), \quad \overline{A} = (A - BK)$$

Consider the usual Lyapunov function

$$V(t) = x^T(t) P x(t)$$

$$\dot{V}(t) = x^T(t)\left[\overline{A}^T P + P\overline{A}\right]x(t) - 2x^T(t)PBK_2 \operatorname{sgn}(x(t))$$

Denoting $Q = -\left[\overline{A}^T P + P\overline{A}\right]$

$$\dot{V}(t) = -x^T(t)Qx(t) - 2x^T(t)PBK_2 \operatorname{sgn}(x(t))$$

$$= -\lambda_{\min}(Q)x^T(t)x(t) - 2x^T(t)PBK_2 \operatorname{sgn}(x(t))$$

$$\leq -\left(\frac{\lambda_{\min}(Q)}{\lambda_{\min}(P)}\right)V(t) - 2x^T(t)PBK_2 \operatorname{sgn}(x(t))$$

In order to prove that the system will attain the origin within a finite time, the second term, $\left(-2x^T(t)PBK_2 \operatorname{sgn}(x(t))\right)$ should be negative-definite. We will show by a simple generic second order counter-example that this term is indeed, sign-indefinite.

$$-2x^T(t)PBK_2 \operatorname{sgn}(x(t)) = -2x^T(t)\tilde{P} \operatorname{sgn}(x(t))$$

We note that any singular matrix of dimension two, can be written as

$$\tilde{P} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ \alpha a & \alpha b \end{bmatrix}$$

(This assertion can be proved easily. Since $\tilde{P}$ is assumed to be singular, $\det \tilde{P} = ad - cb = 0$. Since $\exists \alpha \in \Re$, such that $c = \alpha a$, it naturally follows that $d = \alpha b$)

$$-\boldsymbol{x}^T(t)\boldsymbol{PBK}_2\,\mathrm{sgn}(\boldsymbol{x}(t)) = -\boldsymbol{x}^T(t)\begin{bmatrix} a & b \\ \alpha a & \alpha b \end{bmatrix}\mathrm{sgn}(\boldsymbol{x}(t))$$

Since

$\mathrm{sgn}(\boldsymbol{x}(t)) = \left[\mathrm{sgn}(x_1(t)); \mathrm{sgn}(x_2(t))\right]$,

$x_i\ \mathrm{sgn}(x_i(t)) = |x_i(t)|$ and

$x_i\ \mathrm{sgn}(x_j(t)) = |x_i(t)|\,\mathrm{sgn}(x_i(t))\,\mathrm{sgn}(x_j(t))$

$$-\boldsymbol{x}^T(t)\boldsymbol{PBK}_2\,\mathrm{sgn}(\boldsymbol{x}(t)) = -(a+b)\left[|x_1(t)| + \alpha|x_2(t)|\right]\left[1+\mathrm{sgn}(x_1(t))\mathrm{sgn}(x_2(t))\right]$$

In the above equation, $\left[1+\mathrm{sgn}(x_1(t))\mathrm{sgn}(x_2(t))\right] \geq 0$ (in fact, it can take the values 0,1,2).

$-\boldsymbol{x}^T(t)\boldsymbol{PBK}_2\,\mathrm{sgn}(\boldsymbol{x}(t)) \geq 0$ under the following two conditions,

$$(a+b) > 0,\ \alpha < 0,\ |x_1(t)| < |x_2(t)| \text{ for some } t > 0$$

$$(a+b) < 0,\ \alpha > 0$$

and $-\boldsymbol{x}^T(t)\boldsymbol{PBK}_2\,\mathrm{sgn}(\boldsymbol{x}(t)) \leq 0$ under the following two conditions:

$$(a+b) > 0,\ \alpha > 0$$

$$(a+b) < 0,\ \alpha < 0,\ |x_1(t)| < |x_2(t)| \text{ for some } t > 0$$

Hence, $-2\boldsymbol{x}^T(t)\tilde{\boldsymbol{P}}\,\mathrm{sgn}(\boldsymbol{x}(t))$ can not be guaranteed to be negative, an essential pre-requisite to prove finite time stability. The reason for this inability is of course, the fact that $\boldsymbol{B}$ is not full rank. ∇

In fact, we earlier exploited the spirit behind Theorem 5.4 to prove the reaching condition for the system (5.26). To see this more clearly, let us revisit the equations (5.21)-(5.23) for the uncertain system.

$$\dot{\overline{\boldsymbol{w}}}(t) = \boldsymbol{A}_{11}\overline{\boldsymbol{w}}(t) + \boldsymbol{A}_{12}\boldsymbol{w}_1(t) + \boldsymbol{T}_1\boldsymbol{f}(\boldsymbol{x},t)$$

$$\dot{\boldsymbol{w}}_1(t) = \boldsymbol{A}_{21}\overline{\boldsymbol{w}}(t) + \boldsymbol{A}_{22}\boldsymbol{w}_1(t) + \boldsymbol{B}_1\boldsymbol{u}(t) + \boldsymbol{T}_2\boldsymbol{f}(\boldsymbol{x},t)$$

Since the matrix B_1, is known to be full rank, any linear combination of m states of the system can be driven to and maintained at zero. Hence, we construct a m-dimensional manifold ('switching function') as

$$s(t) = K\bar{w}(t) + w_1(t)$$

In order to drive and maintain these m states to zero (*i.e.*, sliding mode), we require that

$$s(t) = 0, \ \dot{s}(t) = 0$$

$$\begin{aligned}\dot{s}(t) = &\left[K(A_{11} - A_{12}K) + (A_{21} - A_{22}K)\right]\bar{w}(t) + (KA_{12} + A_{22})s(t) \\ &+ B_1 u(t) + (KT_1 + T_2)f(x,t)\end{aligned}$$

Since B_1 is of rank m, we exploit this and design a controller

$$\begin{aligned}u(t) = &-(B_1)^{-1}\left[K(A_{11} - A_{12}K) + (A_{21} - A_{22}K)\right]\bar{w}(t) + (KA_{12} + A_{22})s(t) \\ &+ (-B_1)^{-1}\left[(\ k|s(t)| + d(t)\ )\ \mathrm{sgn}(s(t))\right]\end{aligned}$$

This of course, assures sliding mode and the reader is referred to the proof of Theorem 5.1, for further details. The important message is the following: When the number of control inputs is less than the number of states, *i.e.*, $m < n$, the best we can achieve with the given physical system is to drive a chosen m dimensional subspace of the system to the origin within a finite time and maintain it there for ever. Of course, we can assure asymptotic stability for the signals corresponding to the remaining $(n-m)$ dimensional manifold, if the original system is controllable. Since we have an uncertain system, there is also the additional challenge of having to overcome the effect of uncertainties.

The beauty of the sliding mode control is that we achieve these twin objectives via design of our controller (5.24). The choice to be made then, is the design of an appropriate m dimensional subspace. There are two scenarios: one, we are free to chose any m-dimensional subspace. In such a case, we choose the subspace to guarantee that the nominal system in the sliding mode is asymptotically stable. This determines the choice of K, and hence $C = [K \ \ I_m]T$. We then design a controller to drive the chosen subspace

with the desired speed to origin (*i.e.*, to attain sliding mode). Another scenario is where the switching function is available in the form of a performance objective. For example, consider the case where the objective is to reduce the norm of a given linear combination of the state deviations. In this case, $\boldsymbol{C}$ is specified and $\boldsymbol{K}$ is obtained via the appropriate partition of $\boldsymbol{CT}^{-1}$. Then, the conditions that we have developed in this chapter aid us to ascertain the feasibility of meeting such a requirement. Of course, a more useful constraint would consider the combination of individual state norms, rather than a norm of a linear combination of the states. This aspect merits further attention.

5.6 Constraints on the Control Input

When the initial condition is known, we can find an upper bound on the norm of the control input. This is important as in many cases of practical interest, there are hard constraints on the norm of the control input $\boldsymbol{u}(t)$ (for example, in chemical engineering process applications, the control valve as hard bounds of 0%-100% valve opening. There may also be safety considerations).

First, we get an upper bound on the norm of the control input. We have the control law

$$\boldsymbol{u}(t) = -\left(\boldsymbol{C}\overline{\boldsymbol{B}}\right)^{-1}\left[\boldsymbol{CAz}(t) + k|s(t)|^{\alpha}\,\mathrm{sgn}[s(t)] + d(t)\mathrm{sgn}[s(t)]\right] \tag{5.73}$$

For the sake of simplicity, we write

$$\boldsymbol{u}(t) = \boldsymbol{u}_{cont}(t) + \boldsymbol{u}_{discont}(t) \tag{5.74}$$

where

$$\boldsymbol{u}_{cont}(t) = -\left(\boldsymbol{C}\overline{\boldsymbol{B}}\right)^{-1}\left[\boldsymbol{CAz}(t)\right] \tag{5.75}$$

$$\boldsymbol{u}_{discont}(t) = -\left(\boldsymbol{C}\overline{\boldsymbol{B}}\right)^{-1}\left[k|s(t)|^{\alpha}\,\mathrm{sgn}[s(t)] + d(t)\mathrm{sgn}[s(t)]\right] \tag{5.76}$$

Using (5.7), (5.11) and (5.19), (5.75) can be written as

$$u_{cont}(t) = -(B_I)^{-1}[K \;\; I\mathrm{m}]\;\; TAT^{-1}\left[\begin{bmatrix} I\mathrm{n-m} \\ -K \end{bmatrix}\overline{w}(t) + \begin{bmatrix} 0\mathrm{n-m} \\ s(t) \end{bmatrix}\right] \qquad (5.77)$$

$$u_{cont}(t) = -(B_I)^{-1}\left[K(A_{11} - A_{12}K) + (A_{21} - A_{22}K)\right]\overline{w}(t) \;\; -(B_I)^{-1}\left[KA_{12} + A_{22}\right]s(t) \qquad (5.78)$$

Define:

$$U_{R1} := -(B_I)^{-1}\left[K(A_{11} - A_{12}K) + (A_{21} - A_{22}K)\right],\; U_{R2} := -(B_I)^{-1}\left[KA_{12} - A_{22}\right] \qquad (5.79)$$

Then it follows that

$$|u(t)| \le \|U_{R1}\|\|\overline{w}(t)| + \|U_{R2}\|\|s(t)| + \left\|(B_I)^{-1}\right\|\left[k|s(t)|^{\alpha} + d(t)\right]\mathrm{sgn}[s(t)] \qquad (5.80)$$

From Theorem 5.1, we have that

$$|s(t)| \le \left[|s(0)| + \frac{\sqrt{2\varepsilon}}{k}\right]e^{-\frac{k}{2}t} - \frac{\sqrt{2\varepsilon}}{k}$$

and hence, $\max_{t\ge 0}|s(t)| = |s(0)|$. Also noting that $\|\mathrm{sgn}[s(t)]\| \le \sqrt{m}$, and $d(t) \le d(0)$, and utilizing the previously developed expression for $|\overline{w}_{\max}(t)|$, we have

$$|u(t)| \le \|U_{R1}\|\|\overline{w}_{\max}(t)| + \|U_{R2}\|\|s(0)| + \sqrt{m}\left\|(B_I)^{-1}\right\|\left[k|s(0)|^{\alpha} + d(0)\right] \qquad (5.81)$$

By focusing attention on the above equations, it is clear that there is a price to be paid for placing the poles of the nominal system in the sliding mode via feedback control matrix K arbitrarily far from the imaginary axis. The expression (5.81) gives a practical way of balancing fast response to allowable values of control signals. Unfortunately, due to the complexity of the controller and the stability conditions, it is not possible to develop a systematic way of enforcing a given controller constraint. However, for the interested reader we mention that such approaches have been studied for simple state feedback control, via Linear Matrix Inequalities based approaches [see Niculescu *et al.,* 1997].

In some applications, it is more advantageous to have an expression for the upper bound on the norm of the individual control inputs. For example, consider the well studied case of control of exothermic CSTR where temperature and concentration are controlled by manipulating the inlet concentration and temperature of the feed , and cooling

water temperature. Obviously, there are different constraints on these manipulated variables. We will get an expression for the upper bound on the individual control inputs. Very little further work is required, and we will exploit our previous developments.

For any matrix R, define: R_i:= ith row of R.

Then it simply follows that

$$|u_i(t)| \le \left|(U_{R1})_i\right| \left\|\overline{w}_{\max}(t)\right| + \left|(U_{R2})_i\right| \left\|s(0)\right| + \sqrt{m} \left\|(B_I^{-1})_i\right\| \left[k|s(0)|^{\alpha} + d(0)\right] \quad (5.82)$$

An additional advantage of the expressions (5.81)-(5.82) is that, given a priori upper bounds on the control signals (for example, a pneumatic valve can only take values between 0% - 100%), they can be used to get an estimate (though conservative) of the region of attraction. Towards this we note that $|z(0)| = |x(0)|$ since $u(\tau) = 0,\ \tau < 0$, and hence $|\overline{w}(0)| = |T_1 z(0)| = |T_1 x(0)|$. Then from (5.81)-(5.82), we can get an upper bound on $|x(0)|$.

5.7 Extension to the Case of Multiple Delays

The control scheme presented in this chapter can be extended in a strightforward fashion to the case of multiple known delays. Consider the uncertain input-delay system

$$\dot{x}(t) = Ax(t) + Bu(t) + \sum_{k=1}^{l} B_k u(t - h_k) + f(x,t) \quad (5.83)$$

To convert this into a system without delay, consider the transformation

$$z(t) = x(t) + \sum_{k=1}^{l} \left[\int_{t-h_k}^{t} e^{A(t-h_k-\tau)} B_k u(\tau) d\tau\right] \quad (5.84)$$

Then following the arguments in Appendix D, it is straightforward to show that

$$\dot{z}(t) = Az(t) + \overline{B}u(t) + f(x,t) \quad (5.85)$$

where $\overline{B} = B + \sum_{k=1}^{l} B_k e^{-Ah_k}$. Of course, (5.85) is of the same form as (5.4).

5.8 Illustrative Example

Consider the uncertain input delay system described by (5.1) where

$$A=\begin{bmatrix}-2 & 1 & 0\\ 0 & -1.1 & 0.51\\ 0 & 0 & 0.8\end{bmatrix},\quad B=\begin{bmatrix}0\\1\\1\end{bmatrix},\quad B_d=\begin{bmatrix}0\\0\\1\end{bmatrix},\quad \text{and}\quad h=0.5$$

The uncertainty is constrained to be such that $|f(x,t)|\le\beta|x(t)|$ where $\beta=0.25$. In particular the following uncertainty is chosen for the purpose of simulation.

$$f(x(t),t)=(0.2+\sin t)\begin{bmatrix}0.1 & 0.03 & -0.02\\ -0.08 & 0.05 & 0.0\\ -0.03 & 0.12 & 0.15\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix}$$

The initial condition is specified as $x(0)=[1\ 1\ 1]^T$.

The design objective is to asymptotically stabilize the uncertain system. The homogeneous part of the system is unstable because A has a positive eigenvalue $\lambda=0.8$. After the transformation (5.3), the system equation (5.4) is realized with $\bar{B}=[0.0975\quad 0.7147\quad 1.6703]^T$ and it is verified that $\text{rank}[A,\bar{B}]=3$. The transformation matrix T is given by

$$T=\begin{bmatrix}1 & 0 & -0.0583\\ 0 & 1 & -0.4279\\ 0 & 0 & 1\end{bmatrix},\quad TAT^{-1}=\begin{bmatrix}-2 & 1 & 0.2645\\ 0 & -1.1 & -0.3030\\ 0 & 0 & 0.8\end{bmatrix}$$

and from equation (5.7), $B_1=1.6703$. The poles in sliding mode for the nominal system are assigned at -1 and -2. This is achieved for $K=[0\quad 0.3301]$. From (5.19), it follows that $C=[0\quad 0.3301\quad 0.8588]$. The controller parameters are chosen to be $k=2$ and $\varepsilon=10^{-4}$. Finally, we set $\alpha=0.5$. To guarantee asymptotic stability in the sliding mode, the Lyapunov matrix is chosen as $P=0.245I_{2\times 2}$ and this satisfies (5.46) (In the stability condition (5.46), LHS = 0.6337 and RHS = 0.8231).

The simulation results are presented in Figures 5.1-5.4. Figure 5.3 shows the state norm of the transformed system (5.4) as a function of time, confirming the asymptotic

stability of the transformed system. Figure 5.1 shows that the original system also is asymptotically stable as predicted by the theory. The switching function and its evolution towards the sliding mode is shown in Figure 5.2. The sliding mode is reached rather quickly at around $t = 0.909$. Figure 5.4, shows the evolution of controller output and some chattering is observed in the sliding mode (as the discontinuous control is given by $d(t) = \beta \|C\| |x(t)| + \varepsilon$). The level of chattering continuously decreases to ε as the norm of the states decay to zero. Perturbation compensation will further minimize this problem.

5.8 Conclusion

This chapter deals with the robust stabilization for a class of input delay systems with uncertain states via sliding mode control approach. By a simple transformation, the original system is converted into a system without any input delay and this system is controlled. The asymptotic stability of the transformed system guarantees the asymptotic stability of the original system. By a proper choice of the switching function and the controller parameters, the reaching condition and the asymptotic stability are assured. The problem of chattering is discussed and promising ways to alleviate this problem are discussed. Further work will focus on applications to systems with input saturation and uncertainty in the time delay. It should also be interesting to compare the performance of the sliding mode approach with other conventional methods for time-delay compensation.

Nomenclature for Chapter 5

A	State matrix	(dimension $n \times n$)
B	Control matrix	(dimension $n \times m$)
B_d	Delayed control matrix	(dimension $n \times m$)
C	Switching function matrix	(dimension $m \times n$)
$d(t)$	Controller parameter	(dimension 1×1)
$f(x(t),t)$	State dependent perturbation	(dimension $n \times 1$)

h	Delay	(dimension 1×1)
K	$m\times(n-m)$ matrix	
$s(t)$	Switching function	(dimension $m\times1$)
T	Transformation matrix	(dimension $n\times n$)
$u(t)$	Control input	(dimension $m\times1$)
$x(t)$	State	(dimension $n\times1$)
$z(t)$	State of the transformed system	(dimension $n\times1$)
$w(t)$	Transformed state	(dimension $n\times1$)
$\overline{w}(t)$	Transformed state	(dimension $(n-m)\times1$)
$w_1(t)$	Transformed state	(dimension $m\times1$)
$V(t)$	Lyapunov Function	(dimension 1×1)
P	Lyapunov function matrix	(dimension $(n-m)\times(n-m)$)
$\hat{P}$	Lyapunov function matrix	(dimension $(n-m)\times(n-m)$)
$\lvert\cdot\rvert$	Scalar/vector norm	(dimension 1×1)
$\lVert\cdot\rVert$	Matrix norm	(dimension 1×1)

Greek Letters

α	controller tuning parameter
β	Bound on state dependent uncertainty
δ	Razhamikhin constant
ε	controller tuning parameter
θ	time index
$\phi(\theta)$	Initial condition

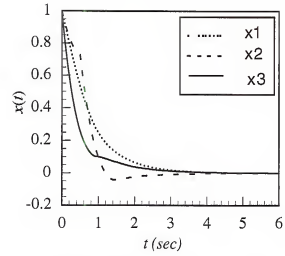

Figure 5.1 Evolution of the states

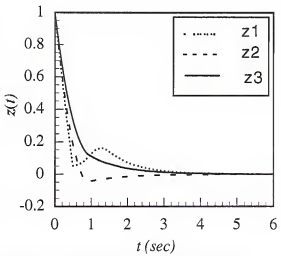

Figure 2. Evolution of the transformed states

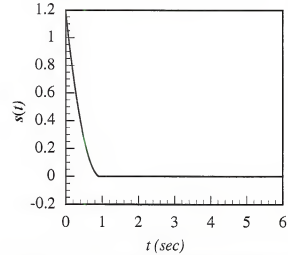

Figure 5.3. Evolution of the switching function

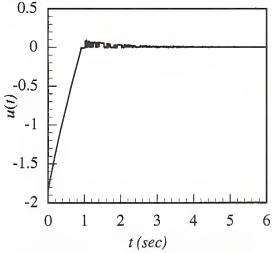

Figure 5.4 Evolution of the control input

CHAPTER 6
FUTURE WORK

6.1 Robust Stabilization of Uncertain Input-Delay Systems-A Linear Matrix Inequality Approach

The problem of robustly stabilizing an uncertain input-delay system via sliding mode control was considered in Chapter 5. In this section, we consider a system with time varying input-delay and uncertain states. Then, we consider the following questions of interest: (1) What is the maximum delay that the system can tolerate and still be robustly stable?, (2) What is the corresponding stabilizing state feedback?, (3) Given a constraint on the controller, is it feasible to stabilize the system and (4) What is the maximum L_2 gain for the system?. We approach this problem via *Linear Matrix Inequalities* (LMI) approach. The work in this section is motivated by and in fact is an extension of the previous works of Niculescu *et al.,* [1997].

Preliminaries and problem formulation

Consider the following uncertain time delay state space system

$$\dot{x}(t) = Ax(t) + B_0u(t) + B_1u(t - \tau(t)) + f(x(t),t) \tag{6.1.1a}$$

Where the uncertainty is bounded by

$$\|f(x(t),t)\| \le \beta_0\|x(t)\| \tag{6.1.b}$$

the time delay is in the range

$$0 < \tau(t) \le \tau^* \tag{6.1.1c}$$

and the controller is given by the state feed back

$$u(t) = Kx(t) \tag{6.1.2a}$$

Given this system, the first objective is to find a stabilizing static feedback controller gain, K. Another goal is find the maximum value of τ^* for which the close loop system is stable. In addition, the problem can be formulated taking into consideration constraints on the control effort, that is

$$\max_{t\geq 0}\|u(t)\| \leq \mu, \quad \mu > 0 \tag{6.1.2b}$$

Assumption 6.1.1.

The pair $(A, B_0 + B_1)$ is stabilizable.

Assumption 1, implies the closed loop system, which is free of delays, can be stabilized.

Controller Design

Let $x(t)$, $t > 0$ be the solution of (6.1.1). Since $x(t)$ is continuously differentiable for $t > 0$, the following equation holds (Hale and Lunel, 1993)

$$\begin{aligned} x(t-\tau(t)) &= x(t) - \int_{-\tau(t)}^{0} \dot{x}(t+\theta)d\theta \\ &= x(t) - \int_{-\tau(t)}^{0} Ax(t+\theta) + B_0Kx(t+\theta) + B_1Kx(t-\tau(t)+\theta) + \\ &\qquad f(x(t+\theta))d\theta, \quad t \geq \tau \end{aligned} \tag{6.1.3}$$

Substituting (6.1.3) into the (6.1.2a), to obtain the delayed input, and then substituting this into the system equation (6.1.1a), letting $B = B_0 + B_1$ gives

$$\begin{aligned} \dot{x}(t) = (A+BK)x(t) - B_1K \int_{-\tau(t)}^{0} & Ax(t+\theta) + B_0Kx(t+\theta) + \\ & B_1Kx(t-\tau(t+\theta)+\theta) + f(t+\theta)d\theta + f(t) \end{aligned} \tag{6.1.4}$$

Which is valid for arbitrary initial data in the range $\left[-2\tau^*, 0\right]$. It can be shown that if the zero solution of (6.1.4) is asymptotically stable then the zero solution of (6.1.1a) is also asymptotically stable [Hale and Lunel, 1993].

Theorem 6.1.1

Consider the uncertain input-delay system given by (6.1.1a-6.1.1c) with known initial condition $\boldsymbol{x}(0)$ under the control action given by (6.1.2a-6.1.2b) with tuning parameter $k_0(\boldsymbol{P})$.

If there exists symmetric and positive-definite matrices $\boldsymbol{Q} \in \mathcal{R}^{n\times n}$ and $N \in \mathcal{R}^{n\times n}$, a matrix $W \in \mathcal{R}^{m\times n}$, and positive scalars $\{\beta_i,\ i=1,\cdots,5\}$ and $\{\mu_i,\ i=1,2\}$ such that the following LMI's, (6.1.5-6.1.12), hold:

$$\begin{bmatrix} \frac{1}{\tau^*}\left[\boldsymbol{Q}\boldsymbol{A}^{\mathrm{T}}+\boldsymbol{A}\boldsymbol{Q}+\boldsymbol{B}W+W^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}+\beta_5 I+N\right]+ & \beta_1 W & \beta_1 W \\ \left[\beta_1+\beta_2+\beta_3\left[k_0(\boldsymbol{P})\beta_0\right]+\beta_4\right] & & \\ W^{\mathrm{T}}\boldsymbol{B}_1^{\mathrm{T}} & -\frac{1}{3}\boldsymbol{Q} & \boldsymbol{0} \\ W^{\mathrm{T}}\boldsymbol{B}_1^{\mathrm{T}} & \boldsymbol{0} & -\beta_3\boldsymbol{Q} \end{bmatrix} < 0 \tag{6.1.5}$$

$$-\beta_1\boldsymbol{Q}+\boldsymbol{A}\boldsymbol{Q}\boldsymbol{A}^{\mathrm{T}} \le 0 \tag{6.1.6}$$

$$\beta_2\begin{bmatrix} -\boldsymbol{Q} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}+\begin{bmatrix} \boldsymbol{0} & \boldsymbol{B}_1 W \\ W^{\mathrm{T}}\boldsymbol{B}_1^{\mathrm{T}} & -\boldsymbol{Q} \end{bmatrix} \le 0 \tag{6.1.7}$$

$$\begin{bmatrix} -N & \boldsymbol{Q} \\ \boldsymbol{Q} & -\beta_0\beta_5^{-1}\boldsymbol{I} \end{bmatrix} \le 0 \tag{6.1.8}$$

$$\mu_1 \boldsymbol{I} \le \boldsymbol{Q} \le \mu_1 k_0(\boldsymbol{P})\boldsymbol{I} \tag{6.1.9}$$

$$\begin{bmatrix} -1 & \boldsymbol{x}(0)^{\mathrm{T}} \\ \boldsymbol{x}(0) & -\boldsymbol{Q} \end{bmatrix} \le 0 \tag{6.1.10}$$

$$\begin{bmatrix} -\boldsymbol{Q} & W^{\mathrm{T}} \\ W & -\mu^2\boldsymbol{I} \end{bmatrix} \le 0 \tag{6.1.11}$$

$$\beta_4\begin{bmatrix} -\boldsymbol{Q} & 0 \\ 0 & \boldsymbol{0} \end{bmatrix}+\begin{bmatrix} 0 & \boldsymbol{B}_0 W \\ W^{\mathrm{T}}\boldsymbol{B}_0^{\mathrm{T}} & -\boldsymbol{Q} \end{bmatrix} \le 0 \tag{6.1.12}$$

then the system is closed-loop asymptotically stable via a finite dimensional controller of

the form $u(t) = Kx(t)$, for all input delays satisfying $0 < \tau(t) \le \tau^*$. Furthermore, the corresponding input is given by

$$u(t) = WQ^{-1}x(t) = Kx(t) \tag{6.1.13}$$

Proof.: The proof of Theorem 1 is based on obtaining sufficient stability conditions for the system given by (6.1.4). This system still contains the specified norm bounded perturbations (6.1.1b), and given constraint on the control input (6.1.2b). The stability requirements are derived using the Lyapunov-Razhamikhin approach. By choosing the following quadratic Lyapunov function

$$V(x(t)) = x(t)^{\mathrm{T}} Px(t) \tag{6.1.14}$$

where P is symmetric, positive definite and an element of $\mathcal{R}^{n\times n}$, all that is need is to show $\dot{V}(x(t)) \le 0$, for $t > 0$ to assure asymptotic stability under the perturbations and constraints. This choice of Lyapunov function has several advantages. One advantage is that if there is an asymptotically stabilizing controller, then

$$\varepsilon = \left\{ x(t) \in \mathcal{R}^{n} \middle| x(t)^{\mathrm{T}} Px(t) \le 1 \right\} \tag{6.1.15}$$

is a holdable ellipsoid. This characterization also makes it possible to impose LMI based conditions to constrain the control input.

In the proof of Theorem 6.1.1, it is useful to impose another restriction on the system. Namely, for any positive number $\delta > 1$, the following holds

$$V(x(\theta)) < \delta V(x(t)), \qquad t - 2\tau \le \theta \le t \tag{6.1.16}$$

This implies that

$$\|x(\theta)\|^2 < \delta k(P)\|x(t)\|^2 \tag{6.1.17}$$

This comes from the fact that the Lyapunov function, (6.1.14), can be bounded by

$$\lambda_{\min}(P)\|x(t)\|^2 \le V(x(t)) \le \lambda_{\max}(P)\|x(t)\|^2 \tag{6.1.18}$$

Considering asymptotically stable systems, it is desired that

$$\|x(\theta)\|^2 < \delta\|x(t)\|^2, \quad \delta > 1, \quad t - 2\tau \le \theta \le t \tag{6.1.19}$$

so that the states decay to zero. Combining (6.1.18) and (6.1.19) gives

$$\lambda_{\min}(\boldsymbol{P})\|x(\theta)\|^2 < \delta\lambda_{\max}(\boldsymbol{P})\|x(t)\|^2 \tag{6.1.20}$$

or

$$\|x(\theta)\|^2 < \delta\frac{\lambda_{\max}(\boldsymbol{P})}{\lambda_{\min}(\boldsymbol{P})}\|x(t)\|^2 = \delta k(\boldsymbol{P})\|x(t)\|^2 \tag{6.1.21}$$

Where $k(\boldsymbol{P})$ is the condition number of $\boldsymbol{P}$ and is equal to $\frac{\lambda_{\max}(\boldsymbol{P})}{\lambda_{\min}(\boldsymbol{P})}$ for symmetric $\boldsymbol{P}$. This restriction is standard in literature.

The time derivative of the Lyapunov function (6.1.14), utilizing the system given by (6.1.4) is

$$\begin{aligned}\dot{V}(x(t)) = x(t)^{\mathrm{T}}\left[(A+\boldsymbol{B}\boldsymbol{K})^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}^{\mathrm{T}}(A+\boldsymbol{B}\boldsymbol{K})\right]x(t) + \\ -2\int_{-\tau(t)}^{0} x(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{B}_1\boldsymbol{K}\boldsymbol{A}x(t+\theta) + x(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{B}_1\boldsymbol{K}\boldsymbol{B}_0\boldsymbol{K}x(t+\theta) + \\ x(t)^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{B}_1\boldsymbol{K})^2 x(t-\tau(t+\theta)+\theta) + x(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{B}_1\boldsymbol{K}f(t+\theta)d\theta + \\ 2x(t)^{\mathrm{T}}\boldsymbol{P}f(t)\end{aligned} \tag{6.1.22}$$

It is desired to bound the individual components of the integral term of (6.1.22) in order to factor out the uncertain time delay and eliminate the integration. This can be done by using the following inequality for any real constant $\beta > 0$ and any symmetric and positive definite matrix $\boldsymbol{S}$, [Su, 1994]

$$-2u^{\mathrm{T}}v \le \beta^{-1}u^{\mathrm{T}}S^{-1}u + \beta v^{\mathrm{T}}Sv \tag{6.1.23}$$

Although the upper bound of (6.1.23) is conservative, this conservatism can be reduced by optimizing over β. For the first term of the integral, consider the following substitutions

$$u^{\mathrm{T}} = x(t)^{\mathrm{T}}(PB_1KA)$$
$$v = x(t+\theta)$$
$$S = P \qquad \text{(6.1.24a-6.1.24c)}$$

Which gives

$$\int_{-\tau(t)}^{0} -2x(t)^{\mathrm{T}}(PB_1KA)x(t+\theta)d\theta$$
$$=$$
$$\int_{-\tau(t)}^{0} -2u^{\mathrm{T}}vd\theta$$
$$\leq$$
$$\int_{-\tau(t)}^{0} \beta_1^{-1}u^{\mathrm{T}}S^{-1}u + \beta_1 v^{\mathrm{T}}Svd\theta \qquad (6.1.25)$$
$$=$$
$$\int_{-\tau(t)}^{0} \beta_1^{-1}x(t)^{\mathrm{T}}(PB_1KA)P^{-1}(PB_1KA)^{\mathrm{T}}x(t) + \beta_1 x(t+\theta)^{\mathrm{T}}Px(t+\theta)$$

Using (6.1.14) and (6.1.16) this becomes

$$\int_{-\tau(t)}^{0} -2x(t)^{\mathrm{T}}(PB_1KA)x(t+\theta)d\theta$$
$$\leq$$
$$x(t)^{\mathrm{T}}\int_{-\tau(t)}^{0} \beta_1^{-1}(PB_1KA)P^{-1}(PB_1KA)^{\mathrm{T}} + \beta_1\delta Pd\theta x(t) \qquad (6.1.26)$$
$$=$$
$$\tau(t)x(t)^{\mathrm{T}}\left[\beta_1^{-1}(PB_1KA)P^{-1}(PB_1KA)^{\mathrm{T}} + \beta_1\delta P\right]x(t)$$

As desired, (6.1.26) is a bound on the integral term that is not in terms of delayed states nor requires integration. Similarly, for the second term of the integral

$$\int_{-\tau(t)}^{0} -2x(t)^{\mathrm{T}}(PB_1KB_0K)x(t+\theta)d\theta$$
$$\leq \qquad (6.1.27)$$
$$\tau(t)x(t)^{\mathrm{T}}\left[\beta_4^{-1}(PB_1KB_0K)P^{-1}(PB_1KB_0K)^{\mathrm{T}} + \beta_4\delta P\right]x(t)$$

For the third term

$$\int_{-\tau(t)}^{0} -2\boldsymbol{x}(t)^{\mathrm{T}}\left(\boldsymbol{P}(\boldsymbol{B}_1\boldsymbol{K})^2\right)\boldsymbol{x}(t-\tau(t+\theta)+\theta)d\theta \leq \tau(t)\boldsymbol{x}(t)^{\mathrm{T}}\left[\beta_2^{-1}\left(\boldsymbol{P}(\boldsymbol{B}_1\boldsymbol{K})^2\right)\boldsymbol{P}^{-1}\left(\boldsymbol{P}(\boldsymbol{B}_1\boldsymbol{K})^2\right)^{\mathrm{T}}+\beta_2\delta\boldsymbol{P}\right]\boldsymbol{x}(t) \tag{6.1.28}$$

The last term in the integral is slightly different. Using (6.1.23) it is bounded by

$$\int_{-\tau(t)}^{0} -2\boldsymbol{x}(t)^{\mathrm{T}}(\boldsymbol{P}\boldsymbol{B}_1\boldsymbol{K})\boldsymbol{f}(t+\theta)d\theta \leq \int_{-\tau(t)}^{0}\beta_3^{-1}\boldsymbol{x}(t)^{\mathrm{T}}(\boldsymbol{P}\boldsymbol{B}_1\boldsymbol{K})\boldsymbol{P}^{-1}(\boldsymbol{P}\boldsymbol{B}_1\boldsymbol{K})^{\mathrm{T}}\boldsymbol{x}(t)+\beta_3\boldsymbol{f}(t+\theta)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{f}(t+\theta)d\theta \tag{6.1.29}$$

Now, the goal is to approximate the term containing the delayed uncertainty with an appropriate bound. From (6.1.18) and the original bound on the uncertainty, (6.1.1b), it is true that

$$\boldsymbol{f}(t+\theta)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{f}(t+\theta) \leq \lambda_{\max}(\boldsymbol{P})\boldsymbol{f}(t+\theta)^{\mathrm{T}}\boldsymbol{f}(t+\theta) \leq \beta_0\lambda_{\max}(\boldsymbol{P})\boldsymbol{x}(t+\theta)^{\mathrm{T}}\boldsymbol{x}(t+\theta) \tag{6.1.30}$$

From (6.1.20) this becomes

$$\boldsymbol{f}(t+\theta)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{f}(t+\theta) \leq \beta_0\lambda_{\max}(\boldsymbol{P})\delta\frac{\lambda_{\max}(\boldsymbol{P})}{\lambda_{\min}(\boldsymbol{P})}\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{x}(t) \tag{6.1.31}$$

Using (6.1.18) and previously mentioned definition of the condition number this becomes

$$\boldsymbol{f}(t+\theta)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{f}(t+\theta) \leq \beta_0\delta\left[\frac{\lambda_{\max}(\boldsymbol{P})}{\lambda_{\min}(\boldsymbol{P})}\right]^2\lambda_{\min}(\boldsymbol{P})\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{x}(t) \leq \beta_0\delta[k(\boldsymbol{P})]^2\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t) \tag{6.1.32}$$

The condition number can be bounded by

$$k(\boldsymbol{P}) \leq k(\boldsymbol{P}_0) \tag{6.1.33}$$

such that

$$\boldsymbol{f}(t+\theta)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{f}(t+\theta) \leq \beta_0\delta[k(\boldsymbol{P}_0)]^2\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t) \tag{6.1.34}$$

Combining (6.1.34) and (6.1.29) gives the bound for the last term of the integral

$$\int_{-\tau(t)}^{0} -2x(t)^{\mathrm{T}}(PB_1K)f(t+\theta)d\theta$$

$$\leq \tag{6.1.35}$$

$$\tau(t)x(t)^{\mathrm{T}}\left[\beta_3^{-1}(PB_1K)P^{-1}(PB_1K)^{\mathrm{T}} + \beta_3\beta_0\delta\left[k(P_0)\right]^2 P\right]x(t)$$

Combining (6.1.26-6.1.28, 6.1.35) and (15) gives

$$\dot{V}(x(t)) \leq x(t)^{\mathrm{T}}\left[M(\tau(t),\delta)\right]x(t) + 2x(t)^{\mathrm{T}}Pf(t) \tag{6.1.36}$$

where

$$M(\tau(t),\delta) = (A+BK)^{\mathrm{T}}P + P^{\mathrm{T}}(A+BK) + \tau(t)\Big[\beta_1^{-1}(PB_1KA)P^{-1}(PB_1KA)^{\mathrm{T}} +$$
$$\beta_2^{-1}\left(P(B_1K)^2\right)P^{-1}\left(P(B_1K)^2\right)^{\mathrm{T}} + \beta_3^{-1}(PB_1K)P^{-1}(PB_1K)^{\mathrm{T}} +$$
$$\beta_4^{-1}(PB_1KB_0K)P^{-1}(PB_1KB_0K)^{\mathrm{T}} + \left[\beta_1+\beta_2+\beta_3\beta_0\left[k(P_0)\right]^2+\beta_4\right]\delta P\Big]$$

(6.1.37)

Asymptotic stability can now be guarantied using the Lyapunov-Razhamikhin approach by showing that the time derivative of the Lyapunov function, (6.1.36), is negative definite subject to the uncertainty constraint (6.1.1b). A necessary condition for (6.1.36) to be negative definite is that $M(\tau(t),\delta)$ be negative definite. Using the continuity property of the eigenvalues of M, it can be argued that if $M(\tau(t),\delta=1)$ is negative definite, then $\exists\delta>1$, sufficiently small such that $M(\tau(t),\delta)$ is negative definite. An LMI based scheme can now be designed to guarantee $M(\tau(t),\delta=1)$ is negative definite. From (6.1.36) it follows that

$$\dot{V}(x(t)) \leq x(t)^{\mathrm{T}}\left[M(\tau(t),\delta=1)\right]x(t) + 2x(t)^{\mathrm{T}}Pf(t) \tag{6.1.38}$$

In addition, the uncertainty constraint can be rewritten as

$$f(t)^{\mathrm{T}}f(t) - x(t)^{\mathrm{T}}(\beta_0 I)x(t) \leq 0 \tag{6.1.39}$$

From (6.1.38), $\dot{V}(x(t))$ is negative definite if

$$\begin{bmatrix} x(t)^{\mathrm{T}} & f^{\mathrm{T}}(t) \end{bmatrix}\begin{bmatrix} M & P \\ P & 0 \end{bmatrix}\begin{bmatrix} x(t) \\ f(t) \end{bmatrix} < 0 \tag{6.1.40}$$

The uncertainty constraint is equivalent to

$$\begin{bmatrix} x(t)^{\mathrm{T}} & f^{\mathrm{T}}(t) \end{bmatrix} \begin{bmatrix} -\beta_0 I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x(t) \\ f(t) \end{bmatrix} \leq 0 \tag{6.1.41}$$

(6.1.40-6.1.41) can be combined into a "single" matrix equation through the s-procedure for quadratic functions and non-strict inequalities [see Boyd et al., 1994]. If there exists a $\beta_5 > 0$, then (6.1.40-6.1.41) are equivalent to the condition

$$\begin{bmatrix} M + \beta_0 \beta_5^{-1} I & P \\ P & -\beta_5^{-1} I \end{bmatrix} < 0 \tag{6.1.42}$$

(6.1.42) can be expanded as

$$M(\tau(t), \delta = 1) + \beta_0 \beta_5^{-1} I + \beta_5 P^2 < 0 \tag{6.1.43}$$

Defining a matrix

$$M_1 := M(\tau(t), \delta = 1) + \beta_0 \beta_5^{-1} I + \beta_5 P^2 \tag{6.1.44}$$

then for any continuous time varying delay, $\tau(t)$, satisfying $0 \leq \tau(t) \leq \tau^*$, there exists an $\alpha_3 > 0$ such that

$$M_1 < 0 \Rightarrow \dot{V}(x(t), t) \leq -\alpha_3 \|x(t)\|^2 \tag{6.1.45}$$

Lemma 6.1.1 gives the sufficient conditions for guaranteeing $M_1 < 0$. Lemma 6.1.2 gives sufficient conditions to ensure that the constraint on the controller, Equation (4b) is satisfied. Lemma 6.1.3 constrains Q such that Equation (6.1.33) holds.

Lemma 6.1.1

There exists a symmetric and positive-definite matrix $P \in \mathcal{R}^{n \times n}$ and a matrix $K \in \mathcal{R}^{m \times n}$ such that $M_1 < 0$, if there exist the symmetric positive-definite matrix $Q \in \mathcal{R}^{n \times n}$ and a matrix $W \in \mathcal{R}^{m \times n}$ satisfying (6.1.5 - 6.1.10).

Proof: From (6.1.37), M_1 is

$$M_1(\tau(t),\delta=1) = (A+BK)^{\mathrm{T}}P + P(A+BK) + \beta_0\beta_5^{-1}I + \beta_5 P^2 + \tau(t)\Big[\beta_1^{-1}(PB_1KA)P^{-1}(PB_1KA)^{\mathrm{T}} + \beta_2^{-1}\Big(P\big(B_1K\big)^2\Big)P^{-1}\Big(P\big(B_1K\big)^2\Big)^{\mathrm{T}} + \beta_3^{-1}(PB_1K)P^{-1}(PB_1K)^{\mathrm{T}} + \beta_4^{-1}(PB_1KB_0K)P^{-1}(PB_1KB_0K)^{\mathrm{T}} + \Big[\beta_1+\beta_2+\beta_3\beta_0\big[k(P_0)\big]^2+\beta_4\Big]P\Big] \tag{6.1.46}$$

The goal is to determine conditions that guarantee $M_1(\tau(t)) < 0$ by satisfying $M_1(\tau^*) < 0$. This condition is not jointly convex in P and K. Fortunately, a change of variables results in an equivalent condition that is an LMI [Bernussout al., 1989]. Multiplying (6.1.46) by Q on both sides and substituting $P = Q^{-1}$ and $K = WQ^{-1}$ results in

$$QM_1Q = QA^{\mathrm{T}} + AQ + BW + W^{\mathrm{T}}B^{\mathrm{T}} + \tau^*\Big[\beta_1^{-1}B_1WQ^{-1}AQA^{\mathrm{T}}Q^{-1}W^{\mathrm{T}}B_1^{\mathrm{T}} + \beta_2^{-1}B_1WQ^{-1}B_1WQ^{-1}W^{\mathrm{T}}B_1^{\mathrm{T}}Q^{-1}W^{\mathrm{T}}B_1^{\mathrm{T}} + \beta_3^{-1}B_1WQ^{-1}W^{\mathrm{T}}B_1^{\mathrm{T}} + \beta_4^{-1}B_1WQ^{-1}B_0WQ^{-1}W^{\mathrm{T}}B_0^{\mathrm{T}}Q^{-1}W^{\mathrm{T}}B_1^{\mathrm{T}} + \Big[\beta_1+\beta_2+\beta_3\beta_0\big[k(P_0)\big]^2+\beta_4\Big]Q\Big] + \beta_0\beta_5^{-1}Q^2 + \beta_5 I \tag{6.1.47}$$

In order to get a tractable sufficient condition, the following constraints are imposed [Niculescu et al., 1997]

$$-\beta_1 Q + AQA^{\mathrm{T}} \le 0 \tag{6.1.48}$$

$$-\beta_2 Q + B_1WQ^{-1}W^{\mathrm{T}}B_1^{\mathrm{T}} \le 0 \tag{6.1.49}$$

In addition, the following constraints are also imposed

$$-\beta_4 Q + B_0WQ^{-1}W^{\mathrm{T}}B_0^{\mathrm{T}} \le 0 \tag{6.1.50}$$

$$-N + \beta_0\beta_5^{-1}Q^2 \le 0 \tag{6.1.51}$$

Note that (6.1.5 - 6.1.6, 6.1.12) represent (6.1.48 - 6.1.51) in LMI form. Substituting (6.1.48 - 6.1.51) into (6.1.47) yields the sufficient condition

$$QM_1Q \le \left[QA^{\mathrm{T}} + AQ + BW + W^{\mathrm{T}}B^{\mathrm{T}} + \beta_5 I + N\right] + \tau^*\left[3B_1WQ^{-1}W^{\mathrm{T}}B_1^{\mathrm{T}} + \beta_3^{-1}B_1WQ^{-1}W^{\mathrm{T}}B_1^{\mathrm{T}} + \left[\beta_1 + \beta_2 + \beta_3\beta_0\left[k(P_0)\right]^2 + \beta_4\right]Q\right] < 0 \quad (6.1.52)$$

(6.1.1b) represents (6.1.52) in LMI form. Hence, if the LMI's given by (6.1.5 - 6.1.8, 6.1.12) are satisfied, it is guaranteed that $M_1 < 0$.

Lemma 6.1.2

Suppose that there exists matrices $W \in \mathcal{R}^{m\times n}$ and $Q = Q^{\mathrm{T}} > 0 \in \mathcal{R}^{n\times n}$ such that $u(t) = WQ^{-1}x(t)$ stabilizes the closed loop system given by (6.1.1 - 6.1.3) with $x(0)$ known, then the following two statements are satisfied.

1. The ellipsoid represented by $\varepsilon = \left\{x \in \mathcal{R}^n \middle| x(t)^{\mathrm{T}}Q^{-1}x(t) \le 1\right\}$ is invariant, and

2. The input constraint given by (6.1.4b) is enforced at all times $t \ge 0$ if the following LMI's are satisfied

$$\begin{bmatrix} -1 & x(0)^{\mathrm{T}} \\ x(0) & -Q \end{bmatrix} \le 0 \quad (6.1.53)$$

$$\begin{bmatrix} -Q & W^{\mathrm{T}} \\ W & -\mu^2 I \end{bmatrix} \le 0 \quad (6.1.54)$$

Proof: An outline of the proof of Lemma 6.1.1 is given in Boyd *et al.*, [1994]. The details are given here for clarity. Since the feedback $u(t) = WQ^{-1}x(t)$ stabilizes the closed loop system the following is true

$$V(x(t),t) < V(x(0),0) \Rightarrow x(0)^{\mathrm{T}}Q^{-1}x(0) \le 1 \Rightarrow x(t)^{\mathrm{T}}Q^{-1}x(t) \le 1$$

Hence, ε is invariant if the LMI given by (6.1.10) holds. In order to constrain the controller, consider

$$\begin{aligned} \max_{t\ge 0}\|u(t)\| &= \max_{t\ge 0}\left\|WQ^{-1}x(t)\right\| \le \mu \\ &= \max_{\varepsilon}\left\|WQ^{-1}x\right\| \\ &\le \max_{\varepsilon}\left\|x^{\mathrm{T}}Q^{-1}W^{\mathrm{T}}WQ^{-1}x\right\|^{1/2} \end{aligned}$$

Using the transformation $\tilde{x} = Q^{-1/2}x$ it is true that

$$\tilde{x}^{\mathrm{T}}(t)\tilde{x}(t) \le 1 \tag{6.1.55}$$

which comes from the invariance of ε. Using this transformation gives

$$\begin{aligned}\max_{t\ge 0}\|u(t)\| &\le \max_{\varepsilon}\left\|\tilde{x}^{\mathrm{T}}(t)Q^{-1/2}W^{\mathrm{T}}WQ^{-1/2}\tilde{x}(t)\right\|^{1/2} \\ &\le \lambda_{\max}\left[Q^{-1/2}W^{\mathrm{T}}WQ^{-1/2}\right]^{1/2}\tilde{x}^{\mathrm{T}}(t)\tilde{x}(t) \\ &\le \lambda_{\max}\left[Q^{-1/2}W^{\mathrm{T}}WQ^{-1/2}\right]^{1/2}\end{aligned} \tag{6.1.56}$$

Hence,

$$\left[Q^{-1/2}W^{\mathrm{T}}WQ^{-1/2}\right] - \mu^2 I \le 0 \quad \Rightarrow \quad \max_{t\ge 0}\|u(t)\| \le \mu, \quad \mu > 0 \tag{6.1.57}$$

This condition can be recast as an LMI

$$\frac{1}{\mu^2}W^{\mathrm{T}}W - QI \le 0 \tag{6.1.58}$$

Equation (49) is a generalized eigenvalue problem if μ is not known. For this problem it is known, and (6.1.58) can be rewritten as

$$\begin{bmatrix} -Q & W^{\mathrm{T}} \\ W & -\mu^2 I \end{bmatrix} \le 0 \tag{6.1.59}$$

Lemma 6.1.3

The symmetric positive definite matrix Q has condition number $k(Q) < k(P_0)$ if there exists $\mu > 0$ such that

$$\mu I \le Q \le \mu k(P_0) I \tag{6.1.60}$$

Proof: Equation(6.1.60) with $k(P_0)$ unknown is an eigenvalue problem [Boyd *et al.*, 1994]. In this case $k(P_0)$ is a tuning parameter. From Equation (6.1.60) it follows that

$$\mu x^{\mathrm{T}}x \le x^{\mathrm{T}}Qx \le \mu k(P_0)x^{\mathrm{T}}x \tag{6.1.61}$$

In addition

$$\lambda_{\min}(Q)x^{\mathrm{T}}x \le x^{\mathrm{T}}Qx \le \lambda_{\max}(Q)x^{\mathrm{T}}x \tag{6.1.62}$$

Equation (6.1.61 - 6.1.62) imply

$$\begin{aligned} \mu &\le \lambda_{\min}(Q) \\ \lambda_{\max}(Q) &\le \mu k(P_0) \end{aligned} \tag{6.1.63}$$

Therefore

$$k(Q) = \frac{\lambda_{\max}(Q)}{\lambda_{\min}(Q)} \le k(P_0) \tag{6.1.64}$$

The goal is to find a static state feedback controller. Note that if the static controller is not feasible through these LMI's, then LMIs that would result through a dynamic controller will also be infeasible.

L_2 Gain

This section considers the problem of finding a state feedback gain, K, such that the L_2 gain

$$\sup_{\|f\|_2=1} \|z\|_2 = \sup_{\|f\|_2 \neq 0} \frac{\|z\|}{\|f\|} \tag{6.1.65}$$

of the closed loop system is less than a specified number γ. Assuming that the bound on the delay, τ^*, is known. Again, the system considered here is

$$\dot{x}(t) = Ax(t) + B_0u(t) + B_1u(t - \tau(t)) + f(x(t), t)$$

with the output given by

$$z(t) = cx(t) \tag{6.1.66}$$

Theorem 6.1.2

The uncertain input-delay system characterized by (6.1.1 - 6.1.4) with known τ^*, has an L_2 gain less than or equal to γ, if the following LMI's are satisfied

$$\begin{bmatrix} \frac{1}{\tau^*}\left[QA^{\mathrm{T}} + AQ + BW + W^{\mathrm{T}}B^{\mathrm{T}} + \beta_5 I + N + c^{\mathrm{T}}c + N_2\right] + & \beta_1 W & \beta_1 W \\ \left[\beta_1 + \beta_2 + \beta_3\left[k_0(P)\beta_0\right] + \beta_4\right] & & \\ W^{\mathrm{T}}B_1^{\mathrm{T}} & -\frac{1}{3}Q & 0 \\ W^{\mathrm{T}}B_1^{\mathrm{T}} & 0 & -\beta_3 Q \end{bmatrix} < 0 \qquad (6.1.67)$$

$$\begin{bmatrix} N_2 & Q \\ Q & \frac{-1}{\gamma^2\beta_0} I \end{bmatrix} \leq 0 \qquad (6.1.68)$$

In addition the LMI's given by (6.1.6 - 6.1.9, 6.1.12) must also be satisfied.

Proof: These LMI's come from arguments similar to those used in proving Theorem 6.1.1. Using the Lyapunov-Razhamikhin approach results in the following conditions

$$\dot{V}(x(t)) + z^{\mathrm{T}}z - \gamma^2 f^{\mathrm{T}} f \leq 0 \qquad (6.1.69)$$

$$\dot{V}(x(t)) + x^{\mathrm{T}}c^{\mathrm{T}}cx - \gamma^2\beta_0 x^{\mathrm{T}}x \leq 0 \qquad (6.1.70)$$

As mentioned, the remaining steps are similar to those show for Theorem 6.1.1, and are therefore omitted.

Future work

The conditions derived in this section are sufficient-only. They give a systematic framework for getting a feasibility answer for the host of questions raised. However, it is imperative to try numerical experiments to see the relative advantages of this method versus the others in the literature. This will form part of the future work.

6.2 Automatic Covariance Resetting for On-Line Recursive Identification

The most well known on-line identification technique is probably the recursive least-squares method (RLS) which has found wide application for the parameter estimation of physical systems. The reasons for its wide spread use are its ease of implementation,

relatively low computational load, and well-known statistical properties. However, problems arise when RLS technique is applied to time-varying systems. The gain of the algorithm decreases monotonically towards zero as a function of time. Hence, after some time the algorithm loses its capability for further tracking any parameter changes. This is a well known phenomena and the approaches proposed for addressing this limitation can be classified in two major groups [see, Ljung, 1987], namely, techniques that use selective-forgetting algorithms, and techniques based on covariance management.

The first group of techniques use the concept of a forgetting factor that selectively weights the data depending on its information content or time of acquisition. The main utility of the forgetting factor is to prevent excessive gain decay, and in addition, to automatically readjust the gain to a high value whenever parameter changes are detected. There are many techniques in the literature. For extensive review and further discussions, see Lung and Gunnarsson [1990], Ramombason *et al.,* [1992], and standard texts such as Goodwin and Sin [1984]. The techniques range from simple exponential forgetting [Goodwin and Sin, 1984], to elaborate schemes such as the directional forgetting strategy of Kulhavy and Kàrny [1984]. Particularly good performance has been observed by the directional-forgetting algorithm of Canudas de Wit and Carillo [1990], which features the concept of a dead zone that can be specified as a function of the measurement-noise variance. Also see, Woo *et al.,* [1995] where, a directional forgetting matrix is employed. Unfortunately, the performance of estimation algorithms with forgetting factors depends strongly on the adequate adjustment of a number of heuristic parameters.

The second group of techniques are those that directly manipulate the parameter-covariance matrix of the RLS algorithm. Such techniques are effective because the gain of the algorithm and the parameter-covariance matrix are directly related. Covariance management is typically done ad hoc, with the assistance of arbitrary premises. The reset limit as well as the new covariance value must be arbitrarily specified by the user. Often these specifications are made invoking strong heuristics, without regard for the level of

process noise or type of parameter-variation dynamics. The arbitrary modifications made to the covariance matrix make it lose its meaning as a statistical covariance. Also, these choices are not obvious in practice.

The high level of heuristic considerations involved in selective forgetting and covariance management is undesirable. Although these two alternative approaches for modifying the RLS algorithm can most often be tuned for successful performance, they involve heuristics or choices which are not obvious to make in practice. This is particularly relevant in the context of adaptive controllers, where the estimation algorithm should not add to the burden of the work required to tune the operation of the controller. Therefore, there is a strong motivation to minimize the number of ad hoc adjustable parameters in RLS estimation algorithms designed for tracking time-varying systems.

In order to implement an on-line identification strategy along with the controller in a process, it is imperative to minimize the number of ad-hoc parameters. With this goal in mind, studies have been done with an improved RLS algorithm for time-varying systems that automatically adjusts its gain matrix in order to track parameter changes Isaksson [1987], [Ramombason *et al.,* 1992]. The new algorithm maintains its capability for tracking parameter changes by estimating on-line the parameter-covariance matrix as well as the measurement-noise variance. Therefore, the scheme has an embedded covariance-estimation algorithm that in principle can characterize the variance of process parameter.

However, there are still some open issues. For example, the performance of the algorithm is suspect when there is process noise. Also, there are no guarantees for convergence. In the ensuing, the standard problem formulation is given and an interesting insight is presented. The problem is the same as considered in Isaksson [1987] and Ramombason *et al.,* [1992]

Let the process be characterized by the time-varying linear regression

$$y(t) = \phi^T(t)\theta(t) + e(t) \tag{6.2.1}$$

where the output $y(t)$ and noise $e(t)$ are scalar signals, and $\phi^T(t) \in \Re^n$ and $\theta(t) \in \Re^n$ are

the regression and parameter vector, respectively. The noise $e(t)$ is assumed to be a zero-mean Gaussian white-noise sequence with variance

$$Ee(t)^2 = R(t) \tag{6.2.2}$$

where E is the expectation operator. The process parameters in (1) may change with time. For example, many physical processes undergo step-wise parameter changes at random times. Other patterns of parameter evolution include ramps, sinusoidal forms, progressive drifts, etc. Often, parameter variations are modeled as a Markovian process, or more specifically, as the random walk

$$\theta(t) = \theta(t-1) + w(t) \tag{6.2.3}$$

where $w(t) \in \Re^n$ is a parameter-innovations vector whose elements are zero-mean white Gaussian noise sequences such that

$$Ew(t)w^T(t) = \tilde{Q}(t) \tag{6.2.4}$$

and

$$Ee(t)w(t) \tag{6.2.5}$$

where the parameter-covariance matrix $\tilde{Q}(t) \in \Re^{n \times n}$ is positive semi-definite. Although the random-walk model (6.2.3) may not be a realistic representation of the type of parameter drifts experienced by a particular process. The random-walk paradigm is nevertheless useful for theoretical purposes.

Least-squares parameter estimation techniques seek to find the estimate $\theta(t)$ that minimizes the quadratic functional

$$\prod(t) = E\sum_{i=1}^{t}\left(y(i) - \varphi^T(i)\theta(i)\right)^2 \tag{6.2.6}$$

The optimal solution to (6.2.6), under the assumption of parameter drift according to the random-walk (6.2.3), is given by the BLUE (Best Linear Unbiased Estimator), a Kalman filter that strikes the best compromise between parameter tracking and noise tolerance

[*cf.* Ljung and Gunnarson, 1990]:

For the case of time-varying systems, the following modified formulation of the BLUE is used:

$$\hat{\theta}(t) = \hat{\theta}(t-1) + L(t)\varepsilon(t) \tag{6.2.7}$$

$$\varepsilon(t) = y(t) - \varphi^T(t)\theta(t) \tag{6.2.8}$$

$$L(t) = \frac{P(t-1)\varphi(t)}{\hat{R}(t) + \varphi^T(t)P(t-1)\varphi(t)} \tag{6.2.9}$$

$$P(t) = P(t-1) - \frac{P(t-1)\varphi(t)\varphi^T(t)P(t-1)}{\hat{R}(t) + \varphi^T(t)P(t-1)\varphi(t)} + \hat{Q}(t) \tag{6.2.10}$$

where the alternative covariance matrix $\hat{Q}(t)$ and variance function $\hat{R}(t)$, obtained by various methods, are used instead of their true unknown counterparts.

The approaches in Ramombason et al., [1992] and Isaksson [1987] consist of estimating $\hat{Q}(t)$ and $\hat{R}(t)$. Towards this, consider the following

$$\varepsilon(t) = \varphi^T(t)w(t) + e(t) \tag{6.2.11}$$

where

$$\hat{w}(t) := \theta(t) - \hat{\theta}(t-1) \tag{6.2.12}$$

Under the assumptions that the random sequence is ergodic, the regressor vector $\varphi(t)$ is deterministic and $E\hat{w}(t)\hat{w}^T(t) = Ew(t)w^T(t)$, it is possible to arrive at

$$\varepsilon(t)^2 = \varphi^T(t)\hat{Q}(t)\varphi(t) + \hat{R}(t) \tag{6.2.13}$$

Since the above can be rearranged in the form of a linear regression, it is then possible to estimate the elements of $\hat{Q}(t)$ and $\hat{R}(t)$. This is a constrained optimization problem, since $\hat{Q}(t)$ has to be positive semi-definite. The convergence properties of the available algorithms have not been proven. Now, we find that a simple substitution,

$$\hat{Q}(t) = \left(\frac{\varepsilon(t)^2 - \hat{R}(t)}{\varphi^T(t)\varphi(t)} \right) I \tag{6.2.14}$$

satisfies (6.2.13). In other words, this solution parametrizes all possible diagonal solutions in terms of a single parameter $\hat{R}(t)$. If $\hat{R}(t)$ can be estimated on-line (or) an approximate value is known this then can be used to estimate an approximate value of $\hat{Q}(t)$ without any need for constrained optimization. Note that for any value of chosen $\hat{R}(t)$, the resulting $\hat{Q}(t)$ is automatically positive semi-definite. This simple, yet elegant solution appears to be new and deserves to be investigated under various scenarios.

6.3 Sliding Mode Control of Bilinear Systems

Another idea is to use the VSC approach for the stabilization of uncertain bilinear systems. Bilinear systems are a special class of nonlinear systems. They are interesting as well as important since various processes in science and engineering can be modeled as bilinear system. Past work in the literature has included optimal control strategies for bilinear systems and the intent of the future research is to compare the performance of the sliding mode control technique against the existing approaches.

Nomenclature for Chapter 6

A	State matrix	(dimension $n \times n$)
B_0	Control matrix	(dimension $n \times m$)
B_1	Delayed control matrix	(dimension $n \times m$)
$e(t)$	Noise	(dimension 1×1)
E	Expectation operator	
$f(x(t),t)$	State dependent perturbation	(dimension $n \times 1$)
$\tau(t)$	Variable delay	(dimension 1×1)
τ^*	Upper bound on variable delay	(dimension 1×1)
K	State feedback matrix	(dimension $m \times n$)

$k_o(\boldsymbol{P})$	Tuning parameter	(dimension 1×1)
$\boldsymbol{u}(t)$	Control input	(dimension $m\times1$)
$\boldsymbol{x}(t)$	State	(dimension $n\times1$)
$V(t)$	Lyapunov function	(dimension 1×1)
$\boldsymbol{P}$	Lyapunov function matrix	(dimension $n\times n$)
$\boldsymbol{P}(t)$	Parameter covariance matrix	(dimension $n\times n$)
N	Positive-definite matrix variable	(dimension $n\times n$)
Q	Positive-definite matrix variable	(dimension $n\times n$)
$\tilde{Q}(t)$	Parameter-covariance matrix	(dimension $n\times n$)
$\hat{Q}(t)$	Estimated covariance matrix	(dimension $n\times n$)
$R(t)$	Variance	(dimension 1×1)
$\hat{R}(t)$	Estimated variance	(dimension 1×1)
W	Matrix variable	(dimension $m\times n$)
$w(t)$	Parameter innovations matrix	(dimension $n\times1$)
$\boldsymbol{x}(0)$	System initial condition	(dimension $n\times1$)

Greek Letters

β_o	Bound on state dependent uncertainty
β_i	Scalar variable
$\varepsilon(t)$	Model prediction error
μ	Upper bound on controller effort
μ_i	Scalar variable
δ	Razhamikhin constant
θ	time index
$\theta(t)$	Parameter vector
$\varphi(t)$	Regression vector

APPENDIX A

In this section, we give a detailed proof of Lemma 3.2. However, before proceeding, we review certain fundamental aspects of the $M-\Delta$ paradigm. For complete details, the reader is referred to Chapter 10 in Zhou *et al.*, [1996]. A standard tool in robustnesss analysis is to exploit the concept of Linear Fractional Transformation (LFT). The following diagram shows an upper LFT, $F_u(M,\Delta)$,

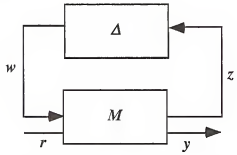

Figure A.1 The classical $M-\Delta$ structure

where Δ is the norm bounded uncertainty and M is a complex matrix that can be partitioned as

$$M=\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

This results in,

$$\begin{pmatrix} z \\ y \end{pmatrix}=\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\begin{pmatrix} w \\ r \end{pmatrix}$$

From the above diagram, it is straightforward to show that

$$y=F_u(M,\Delta)=\left[M_{22}+M_{21}\Delta\left(I-M_{11}\Delta\right)^{-1}M_{12}\right]r$$

In the above equation, M_{22} represents the nominal mapping, and is perturbed by Δ, while M_{11}, M_{12}, and M_{21} reflect the way in which the perturbations affect the nominal map. The control theoretic insight is gained by obeserving that the $F_u(M,\Delta)$ is, in fact, the closed loop transfer function between the input and the output. Obviously, for the upper LFT to be well defined, the $(I - M_{11}\Delta)$ must be invertible. Naturally, this leads to the determinantal condition $\det(I - M_{11}\Delta) \neq 0$.

In the following analysis, with little abuse of notation we will refer to M_{11} as M. Our task is to find M, that is the transfer function between z and w. The SISO transfer function (3.1) with affine uncertainty can be written as shown in (3.3).

$$g(j\omega,q) = \frac{n(j\omega,0) + \sum_{k=1}^{m} n_k(j\omega)q_k(j\omega)}{d(j\omega,0) + \sum_{k=1}^{m} d_k(j\omega)q_k(j\omega)}$$

The aim is to transform the system (3.3) under unity feedback in to a $M-\Delta$ structure by extracting the parametric uncertainties and including them in a diagonal the matrix

$$\Delta = \text{diag}\{q_1, q_2, \ldots, q_m\}$$

For ease of notation, henceforth, we will drop the arguments and denote, $n_0 = n(j\omega,0)$ and $d_0 = d(j\omega,0)$. With these simplifications, we can write

$$g(s) = \frac{y(s)}{u(s)} = \frac{n_0 + \sum_{k=1}^{m} n_k q_k}{d_0 + \sum_{k=1}^{m} d_k q_k}$$

Cross multiplication yields,

$$y = \frac{n_0}{d_0}u + \sum_{k=1}^{m} q_k \left[\frac{n_k}{d_0}u - \frac{d_k}{d_0}y\right]$$

Conceptually, this can be represented by Figure A.2. While for ease of exposition, Figure A.2 shows only one uncertainty, multiple uncertainties can be trivially added to the diagram. Figure A.3 shows the SISO system under unity feedback. Furthermore, in order

to extract the effect of uncertainty, we label the input and output of the uncertainties as z_k and w_k, respectively. Then, by following Figure A.3, yields

$$y = (r - y)\frac{n_0}{d_0} + \sum_{k=1}^{m} w_k$$

$$= \left(\frac{d_0}{n_0 + d_0}\right)\sum_{k=1}^{m} w_k + \left(\frac{n_0}{n_0 + d_0}\right) r$$

$$z_k = (r - y)\frac{n_k}{d_0} - \frac{d_k}{d_0} y$$

$$= r\frac{n_k}{d_0} - \frac{(n_k + d_k)}{d_0} y$$

Substituting for y reults in

$$z_k = \left[\frac{n_k}{d_0} - \frac{(n_k + d_k)n_0}{(n_0 + d_0)d_0}\right] r - \frac{(n_k + d_k)}{(n_0 + d_0)} \sum_{k=1}^{m} w_k$$

Defining M_{ik} as the transfer function from the output of q_k to the input of q_i, it is straightforward to conclude that

$$M_{ik} = -\frac{(n_k + d_k)}{(n_0 + d_0)}$$

Using definitions (3.5) and (3.6) leads to the final result

$$M_{ik}(s) = -\frac{p_k(s)}{p_o(s)}, \quad i = 1, 2, \cdots, m; \quad k = 1, 2, \cdots, m$$

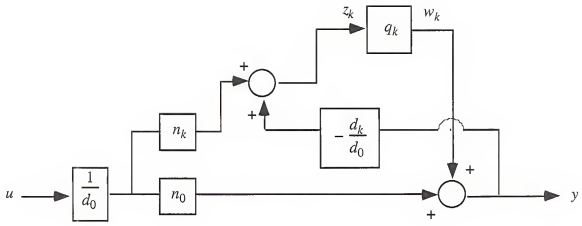

Figure A.2 Block diagram of the open-Loop SISO Transfer function

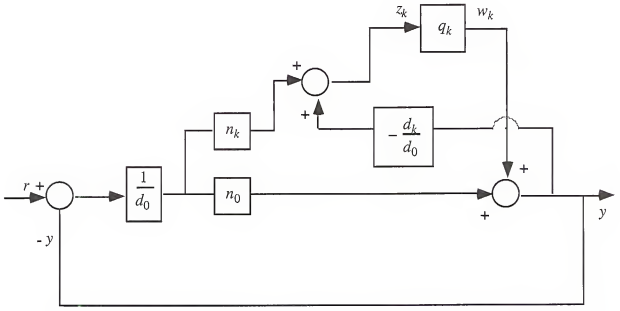

Figure A.3 Block diagram of the closed-Loop SISO Transfer function

APPENDIX B

Proof of Theorem 4.2

From (4.15) we have,

$$\dot{x}(t) = \overline{A}x(t) + \overline{A}_d x(t-h) + f(x(t),t) + f_d(x(t-h),t) \tag{B.1}$$

In order to derive sufficiency conditions, the perturbations can be replaced by their upper bounds as in equation (4.2). Then, following steps similar to those in the proof of Theorem 4.1 in [Mori *et al., 1981*] and using the Comparison Theorem [Hale and Lunel, 1993], it can be shown that system (4.12) is asymptotically stable if (4.13) holds. The details are given below:

The solution $x(t), \forall t > 0$ can be written as

$$x(t) = \exp(\overline{A}t)x(0) + \int_0^t \exp(\overline{A}(t-s))r(s)ds \tag{B.2}$$

where $r(s)$ is given by

$$r(s) = \overline{A}_d x(s-h) + f(x(s),s) + f_d(x(s-h),s) \tag{B.3}$$

Taking the norm on both sides of (B.2) and using (B.3) and the following inequality

$$|\exp(\overline{A}t)| \le \exp(\mu(\overline{A})t), \ \forall t \ge 0 \text{ [Vidyasagar, 1993]}$$

yields

$$|x(t)| \le \exp(\mu(\overline{A})t)x(0) + \int_0^t \exp(\mu(\overline{A})(t-s))|r(s)|ds \tag{B.4}$$

Put $v(t) = |x(t)|$, we can rewrite (B.4) as

$$v(t) \le \exp(\mu(\overline{A})t)v(0) + \int_0^t \exp(\mu(\overline{A})(t-s))|r(s)|ds \tag{B.5}$$

with $v(t) = |\phi(t)|,\ -h \le t < 0$, and $v(0) = v_0$.

Now, consider the integral equation with the deviating argument

$$z(t) = \exp\left(\mu(\overline{A})t\right)v(0 + \int_0^t \exp\left(\mu(\overline{A})(t-s)\right)\left[\left(\left[|\overline{A}_d| + k_d\right]v(s-h)| + k|v(s)|\right)\right]ds \quad \text{(B.6)}$$

with $z(t) = |\phi(t)|,\ -h \le t < 0$.

Of course, (B.6) is the solution of the scalar differential equation

$$\dot{z}(t) = \mu(\overline{A})z(t) + \left[|\overline{A}_d| + k_d\right]z(t-h) + kz(t),\ z(0) = v_0 \quad \text{(B.7)}$$

The solution of the above differential equation uniquely exists and is asymptotically stable if and only if

$$-\left(\mu(\overline{A}) + k\right) > \left[|\overline{A}_d| + k_d\right]$$

$$\mu(\overline{A}) + k + |\overline{A}_d| + k_d < 0 \quad \text{(B.8)}$$

If the condition (B.8) is satisfied, using the comparison theorem with m=1 (See Appendix C), we can write $|x(t)| = v(t) \le z(t)$.

Hence, $\lim_{t\to\infty} z(t) \to 0 \Rightarrow |x(t)| \to 0$. ∇

APPENDIX C

Comparison Theorem

Assume a vector valued function $m(t,s,z): J \times J \times R^m \to R^m, J := [t_0, \infty]$ has the following property.

For any fixed *t, s,*

$$z_1 \le z_2 \Rightarrow m(t,s,z_1) \le m(t,s,z_2)$$

Let $z(t)$ be the solution to the inequality

$$z(t) \le z(0) + \int_{t_o}^{t} m(t,s,z(s-h))ds$$

Then the maximal solution $r(t)$ of $w(t) \le z(0) + \int_{t_o}^{t} m(t,s,w(s-h))ds$ satisfies

$$z(t) \le r(t), t \ge t_o$$

For a proof, see [Laksmikantham and Leela, 1969].

APPENDIX D

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t) + B_d u(t-h) + f(x(t),t) \tag{D.1}$$

Introduce the transformation

$$z(t) = x(t) + \int_{t-h}^{t} e^{A(t-h-\tau)} B_d u(\tau) d\tau \tag{D.2}$$

Taking the time derivative to (D.2), we have

$$\dot{z}(t) = \dot{x}(t) + \frac{d}{dt}\left[\int_{t-h}^{t} e^{A(t-h-\tau)} B_d u(\tau) d\tau\right] + f(x(t),t) \tag{D.3}$$

Using Leibnitz rule for differentiation under the integral sign, we have

$$\frac{d}{dt}\left[\int_{t-h}^{t} e^{A(t-h-\tau)} B_d u(\tau) d\tau\right] = A\int_{t-h}^{t} e^{A(t-h-\tau)} B_d u(\tau) d\tau + e^{-Ah} B_d u(\tau) - B_d u(t-h)$$

Substituting (D.4) into (D.3), having

$$\dot{z}(t) = Az(t) + \left[B + e^{-Ah} B_d\right] u(t) + f(x(t),t)$$

APPENDIX E

If $A \in \Re^{n\times n}$, $\overline{B} \in \Re^{n\times m}$ *is a controllable pair, the a nonsingular matrix* $T \in \Re^{n\times n}$ *can be found such that*

$$TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad T\overline{B} = \begin{bmatrix} 0 \\ B_I \end{bmatrix}$$

then the system $\left[A_{11}, A_{12}\right]$ *is also completely controllable.*

Proof: First of all, we note that controllability (and observability) are preserved under similarity transformation (Kailath, 1980).

Hence, from PBH test, given

$$\text{rank}\left[sI - A, \overline{B}\right] = n \quad \forall s \tag{E.1}$$

it implies that

$$\text{rank}\left[sI - TAT^{-1}, T\overline{B}\right] = n \quad \forall s \tag{E.2}$$

That is no left eigenvector of TAT^{-1} is orthogonal to $T\overline{B}$. This implies that there can not be a non zero row vector $p \in \Re^{n\times 1}$ such that

$$p\left[sI - TAT^{-1}\right] = 0 \quad \forall s \tag{E.3}$$

and

$$p\overline{B} = 0 \quad \forall s \tag{E.4}$$

We partition p as

$$p = \begin{bmatrix} p_{n-m} & p_m \end{bmatrix} \tag{E.5}$$

Using (E.3)-(E.5) in conjunction with (5.11) yields an equivalent condition for controllability.

The system $[A, \overline{B}]$ is controllable, if and only if there exists no non zero row vectors $p_{n-m} \in \Re^{n-m}$, $p_m \in \Re^m$ such that

$$\begin{bmatrix} p_{n-m} & p_m \end{bmatrix} \begin{bmatrix} sI - A_{11} & -A_{12} & 0 \\ -A_{21} & sI - A_{22} & \overline{B} \end{bmatrix} = 0 \quad \forall\, s \tag{E.6}$$

$$p_m \overline{B} = 0 \quad \forall\, s \tag{E.7}$$

Note that since $\overline{B}$ is nonsingular square matrix the only solution for (E.7) is given by $p_m = 0$ Therefore, there exists no row vector $p_{n-m} \in \Re^{(n-m)\times 1}$ such that

$$p_{n-m}\left(sI - A_{11}\right) = 0 \quad \forall\, s \tag{E.8}$$

$$p_{n-m} A_{12} = 0 \quad \forall\, s \tag{E.9}$$

Of course, (E.8) and (E.9) imply that

$$\text{rank}\left[sI - A_{11}, A_{12}\right] = n - m \quad \forall s \tag{E.10}$$

Hence the dynamical subsystem $\left[A_{11}, A_{12}\right]$ is completely controllable.

APPENDIX F

Consider the following scalar differential equation

$$\dot{x}(t) \le -ax(t) - bx^{1/2}(t)$$

with known $x(0)$ *and subject to the constraints*

(1) the constants $a > 0,\ b > 0$

(2) $x(t) > 0$ *and* $\sqrt{x(t)} > 0$

$$\sqrt{x(t)} \le \exp\left(\frac{-a}{2}t\right)\left[\sqrt{x(0)} + \frac{b}{a}\right] - \frac{b}{a}$$

Proof:

By effecting a transformation

$$x(t) = [y(t)]^2 \tag{F.1}$$

(F.1) becomes

$$2y(t)\frac{dy(t)}{dt} \le -ay^2(t) - by(t) \tag{F.2}$$

Since $y(t) > 0$ from the constraints

$$\frac{dy(t)}{dt} \le \left(-\frac{a}{2}\right)y(t) + \left(-\frac{b}{2}\right) \tag{F.3}$$

(F.3) is of the form

$$\frac{dy}{dt} + py \le q \tag{F.4}$$

Hence,

$$y(t)\exp\left(\frac{a}{2}t\right) \le \int\left(\frac{-b}{2}\right)\exp\left(\frac{a}{2}t\right)dt + C$$

$$\leq \frac{-b}{2}\exp\left(\frac{a}{2}t\right)\left(\frac{2}{a}\right)+C$$

$$\leq \frac{-b}{a}\exp\left(\frac{a}{2}t\right)+C \tag{F.5}$$

$$y(t) \leq \frac{-b}{a}+C\exp\left(\frac{-a}{2}t\right) \tag{F.6}$$

$$y(t) \leq \exp\left(\frac{-a}{2}t\right)\left[y(0)+\frac{b}{a}\right]-\frac{b}{a} \tag{F.7}$$

$$\sqrt{x(t)} \leq \exp\left(\frac{-a}{2}t\right)\left[\sqrt{x(0)}+\frac{b}{a}\right]-\frac{b}{a} \tag{F.8}$$

The transformation from (F.2) to (F.3) is valid for the range $0 \leq t < \frac{2}{a}\ln\left[\frac{a\sqrt{x(0)}}{b}+1\right]$.

For all practical purposes, the upper bound can be taken as the time taken for the system to attain zero.

To prove the assertion in the corollary, consider the scalar differential equation

$$\dot{x}(t) < -k\left[x(t)\right]^{0.5(1+\alpha)}, \quad x(t) > 0 \tag{F.9}$$

with known $x(0)$ *under the constraints:*

(1) the constant $k > 0$

(2) $x(t) > 0$ *and* $\sqrt{x(t)} > 0$

Integrating (F.9), it is straightforward to show

$$\frac{\left[x(t)\right]^{-0.5(1+\alpha)+1}}{0.5\times(1-\alpha)} \leq -kt + c_1$$

Substituting for the constant yields

$$x(t) \leq \left[\frac{(1-\alpha)}{2}\left[\frac{2}{(1-\alpha)}\left[x(0)\right]^{0.5(1-\alpha)}-kt\right]\right]^{\left(\frac{2}{1-\alpha}\right)} \tag{F.10}$$

The time to attain $x(t)=0$ is given by

$$t_r \le \frac{2}{k(1-\alpha)}\left[\frac{1}{2}s^T(0)s(0)\right]^{0.5(1-\alpha)} \tag{F.11}$$

and since

$$x(t) = V(t) = \frac{1}{2}s^T(0)s(0) = \frac{1}{2}|s(t)|^2$$

$$|s(t)| \le \sqrt{2}\left[\frac{(1-\alpha)}{2}\left[\frac{2}{(1-\alpha)}[x(0)]^{0.5(1-\alpha)} - kt\right]\right]^{\left(\frac{2}{1-\alpha}\right)}, \quad 0 \le t \le t_r \tag{F.12}$$

APPENDIX G

Consider the following scalar differential equation

$$\dot{x}(t) \le -ax(t) + \left(b\exp(-ct/2) + e\ \exp(-\delta_2 t/2) - d\right)x^{1/2}(t)$$

with the initial condition $x(0)$ *and the following constraints:*

(1) The constants $(a,b,c,d,e) > 0$ *and* $a \neq c$ *and* $a \neq \delta_2$

(2) $x(t) > 0$ *and* $\sqrt{x(t)} > 0$

Then,

$$\sqrt{x(t)} \le \left[\sqrt{x(0)} + \frac{b}{a}\right]\exp\left(-\frac{a}{2}t\right) + \frac{b}{(a-c)}\left[\exp\left(-\frac{c}{2}t\right) - \exp\left(-\frac{a}{2}t\right)\right]$$
$$+ \frac{e}{(a-\delta_2)}\left[\exp\left(-\frac{\delta_2}{2}t\right) - \exp\left(-\frac{a}{2}t\right)\right] - \frac{d}{a}$$

Proof: By effecting a transformation

$$x(t) = [y(t)]^2 \tag{G.1}$$

(G.1) becomes

$$2y(t)\frac{dy(t)}{dt} \le -ay^2(t) + \left(b\exp(-ct/2) + e\ \exp(-\delta_2 t/2) - d\right)y(t) \tag{G.2}$$

Since $y(t) > 0$ from constraints, one can divide both sides by $y(t)$ to get

$$\frac{dy(t)}{dt} \le \left(-\frac{a}{2}\right)y(t) + \left(b/2\exp(-ct/2) + e/2\ \exp(-\delta_2 t/2) - d/2\right) \tag{G.3}$$

(G.3) is of the form

$$\frac{dy}{dt} + py \le q$$

Hence, the solution of (G.3) is given by

$$y(t)\exp\left(\frac{at}{2}\right) \le \int\left(b/2\exp\left(\frac{-ct}{2}\right)+e/2\ \exp\left(\frac{-\delta_2 t}{2}\right)-d/2\right)\exp\left(\frac{at}{2}\right)dt + c$$

$$y(t) \le \left(\frac{b}{a-c}\right)\exp\left(\frac{-ct}{2}\right)+\left(\frac{e}{a-\delta_2}\right)\exp\left(\frac{-\delta_2 t}{2}\right)-\frac{d}{a}+c\exp\left(\frac{-at}{2}\right) \quad \text{(G.4)}$$

solving for the constant c, yields

$$y(t) \le \left(\frac{b}{a-c}\right)\exp\left(\frac{-ct}{2}\right)-\frac{d}{a}+\left(y(0)+\frac{d}{a}-\frac{b}{a-c}-\frac{e}{a-\delta_2}\right)\exp\left(\frac{-at}{2}\right) \quad \text{(G.5)}$$

It is instructive to rewrite (G.5) as

$$\begin{aligned} y(t) \le & \left(y(0)+\frac{d}{a}\right)\exp\left(\frac{-at}{2}\right)+\left(\frac{b}{a-c}\right)\left[\exp\left(\frac{-ct}{2}\right)-\exp\left(\frac{-at}{2}\right)\right] \\ & +\left(\frac{e}{a-\delta_2}\right)\left[\exp\left(\frac{-\delta_2 t}{2}\right)-\exp\left(\frac{-at}{2}\right)\right]-\frac{d}{a} \end{aligned} \quad \text{(G.6)}$$

In (G.6) the first three terms are positive, while the third term is a negative constant. Furthermore, the first term continuously decreases from it's initial value, while the second and terms always have a maxima. As a parallel we remark that the second and third terms in (G.6) look like the response of an under-damped second order system to unit impulse input.

Since $y(t) = \sqrt{x(t)}$, the result follows ∇

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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